

MACROSCOPIC EVOLUTION OF MECHANICAL AND THERMAL ENERGY IN A HARMONIC CHAIN WITH RANDOM FLIP OF VELOCITIES

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ABSTRACT. We consider an unpinned chain of harmonic oscillators with periodic boundary conditions, whose dynamics is perturbed by a random flip of the sign of the velocities. The dynamics conserves the total volume (or elongation) and the total energy of the system. We prove that in a diffusive space-time scaling limit the profiles corresponding to the two conserved quantities converge to the solution of a diffusive system of differential equations. While the elongation follows a simple autonomous linear diffusive equation, the evolution of the energy depends on the gradient of the square of the elongation.

1. INTRODUCTION

Harmonic chains with energy conserving random perturbations of the dynamics have recently received attention in the study of the macroscopic evolution of energy [1, 2, 5, 7, 9, 11]. They provide models that have a non-trivial macroscopic behavior which can be explicitly computed. We consider here the dynamics of an unpinned chain where the velocities of particles can randomly change sign. This random mechanism simulates the collisions with independent *environment* particles of infinite mass. Since the chain is unpinned, the only conserved quantities of the dynamics are the *energy* and the *volume* (or elongation).

Under a diffusive space-time scaling the profile of elongation evolves independently of the energy and follows the linear diffusive equation

$$\partial_t r(t, u) = \frac{1}{2\gamma} \partial_{uu}^2 r(t, u).$$

Here u is the Lagrangian space coordinate of the system. The energy profile can be decomposed into the sum of *mechanical* and *thermal* energy

$$e(t, u) = e_{\text{mech}}(t, u) + e_{\text{thm}}(t, u)$$

where the mechanical energy is given by $e_{\text{mech}}(t, u) = \frac{1}{2} (r(t, u))^2$, while the thermal part $e_{\text{thm}}(t, u)$, that coincides with the temperature profile, evolves following

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the non-linear equation:

$$\partial_t e_{\text{thm}}(t, u) = \frac{1}{4\gamma} \partial_{uu}^2 e_{\text{thm}}(t, u) + \frac{1}{2\gamma} (\partial_u r(t, u))^2.$$

Concerning the distribution of the energy in the frequency modes: the mechanical energy $e_{\text{mech}}(t, u)$ is concentrated on the modes corresponding to the largest wavelength, while the thermal energy $e_{\text{thm}}(t, u)$ is distributed uniformly over all frequencies. Note that $\frac{1}{2\gamma} (\partial_u r(t, u))^2$ is the rate of dissipation of the mechanical energy into thermal energy, which is expected in a non-linear deterministic model that is sufficiently chaotic.

The presence of the non-linearity in the evolution of the energy makes the macroscopic limit non-trivial. Relative entropy methods (as introduced in [12]) identify correctly the limit equation (see [11]), but in order to make them rigorous one needs sharp bounds on higher moments than cannot be controlled by the relative entropy¹. In this sense the proof in [11] is not complete.

We follow here a different approach based on Wigner distributions, successfully applied for systems perturbed by different noise with more conservation laws in [5, 7]. The Wigner distributions permit to control the energy distribution over various frequency modes and provide a natural separation between mechanical and thermal energies. The initial distributions can be random, and the only condition we ask, beside to have definite mean asymptotic profiles of elongation and energy, is that the thermal energy spectrum as a square integrable density. In the macroscopic limit we prove, in some weak sense, that locally the thermal energy spectrum has a constant density equal to the local thermal energy (or temperature). In this sense we prove that the system is, at macroscopic positive times, in local equilibrium, even though it is not at initial time.

The thermalization and the correlation structure of the pinned model have been studied in [8, 9]. Note that in the pinned model only energy is conserved, no mechanical energy is present and consequently the macroscopic evolution is given by a linear heat equation. Of course our method can also be applied directly to the pinned model.

For an anharmonic chain with velocity flip dynamics, the hydrodynamic limit is a difficult non-gradient problem, for the moment still open. The linear response and the existence of the Green-Kubo formulas for the thermal diffusivity have been proven in [3].

2. MICROSCOPIC DYNAMICS

2.1. Periodic chain of oscillators. We are interested in the behavior of a one-dimensional harmonic chain of n oscillators, all of mass 1, when boundary conditions are periodic and n goes to infinity. More precisely, let $R_n = nr$ the total

¹For more details related to these moments bounds, that are still conjectured but not proved (on the contrary to what is claimed in [11]), we refer the reader to an *erratum* which is available online at <http://chercheurs.lille.inria.fr/masimon/erratum-v2.pdf>.

elongation of the system (that will remain fixed in time). The particles will move in the circle $\mathbb{T}(R_n)$ of length R_n . The position of the particle x is denoted by $q_x \in \mathbb{T}(R_n)$ (as an element of \mathbb{R} , q_x is defined only up to an integer multiplicity of R_n), while its velocity (or momentum since all particles have mass 1) is denoted by $p_x \in \mathbb{R}$, where $x \in \{1, \dots, n\}$ is the label.

Each particle x is connected to the particles $x-1$ and $x+1$ by harmonic springs. Moreover, particle n interacts in the same way with particles $n-1$ and 1 (this is the sense of the periodic conditions). The elongation of the spring between $x-1$ and x , can be an arbitrary real number that we call r_x (it can be negative, or also bigger than R_n). Its projection modulo R_n satisfies $[r_x]_{\equiv R_n} = q_x - q_{x-1}$. Consequently, in order to describe the dynamics and its Hamiltonian, we need the set of elongations between particles $\mathbf{r} := \{r_x \in \mathbb{R}, x = 1, \dots, n\}$, where r_1 denotes the elongation of the spring between particle n and particle 1, and it satisfies the constrain $\sum_{x=1}^n r_x = R_n$. Precisely, the interaction between two particles $x-1$ and x is described by the quadratic potential energy $\frac{1}{2}r_x^2$ (for $x = 2, \dots, n$) of a harmonic spring relating the particles. Particle n interacts with particle 1 by the same interaction $\frac{1}{2}r_1^2$.

Let $\mathbb{T} := \mathbb{T}(1)$ be the continuous circle of length 1, and let $\mathbb{T}_n := \mathbb{Z}/n\mathbb{Z} = \{0, \dots, n-1\}$ be the corresponding discrete circle. Since there is a natural translation invariance of the interaction with respect to discrete shifts of the indices, it is useful to consider $x \in \mathbb{T}_n$ in what follows. A configuration of the system is then given by $\{(r_x, p_x) \in \mathbb{R}^2, x \in \mathbb{T}_n\}$, and the configuration space is denoted by $\Omega_n := (\mathbb{R} \times \mathbb{R})^{\mathbb{T}_n}$. The total energy of the system is given by the Hamiltonian:

$$\mathcal{E}_n := \sum_{x \in \mathbb{T}_n} \mathcal{E}_x, \quad \mathcal{E}_x := \frac{p_x^2}{2} + \frac{r_x^2}{2}.$$

2.2. Stochastically perturbed hamiltonian evolution. In addition to the hamiltonian dynamics associated to the harmonic potentials, particles are subject to an interaction with the environment: at independently distributed random poissonian times, the momentum p_x is flipped into $-p_x$. This noise still conserves the total energy \mathcal{E}_n and the total volume R_n . The equations of motion are thus given by

$$\begin{cases} dr_x(t) = n^2(p_x(t) - p_{x-1}(t)) dt \\ dp_x(t) = n^2(r_{x+1}(t) - r_x(t)) dt - 2p_x(t^-) d\mathcal{N}_x(\gamma n^2 t), \end{cases} \quad x \in \mathbb{T}_n. \quad (2.1)$$

Here $\{\mathcal{N}_x(t) ; t \geq 0, x \in \mathbb{T}_n\}$ are n independent Poisson processes of intensity 1, and the constant γ is strictly positive. We have already rescaled time according to the diffusive space-time scaling. The generator of this diffusion can be written as

$$\mathcal{L}_n := n^2 \mathcal{A}_n + n^2 \gamma \mathcal{S}_n,$$

the Liouville operator \mathcal{A}_n being formally given by

$$\mathcal{A}_n = \sum_{x \in \mathbb{T}_n} \left\{ (p_x - p_{x-1}) \frac{\partial}{\partial r_x} + (r_{x+1} - r_x) \frac{\partial}{\partial p_x} \right\},$$

while, for $f : \Omega_n \rightarrow \mathbb{R}$,

$$\mathcal{S}_n f(\mathbf{r}, \mathbf{p}) = \sum_{x \in \mathbb{T}_n} \{f(\mathbf{r}, \mathbf{p}^x) - f(\mathbf{r}, \mathbf{p})\}$$

where \mathbf{p}^x is the configuration that is obtained from \mathbf{p} by reversing the sign of the velocity at site x , namely: $(\mathbf{p}^x)_y = p_y$ if $y \neq x$ and $(\mathbf{p}^x)_x = -p_x$.

We should consider the two conserved quantities $\mathcal{E}_n = \sum_{x \in \mathbb{T}_n} \mathcal{E}_x$ and $R_n = \sum_{x \in \mathbb{T}_n} r_x$, as determined by the initial data (eventually random), and typically they should be proportional to n : $\mathcal{E}_n = ne$, $R_n = nr$. Notice that the dynamics are perfectly well defined also for negative values of R_n .

The system has a family of stationary measures given by the canonical Gibbs distributions

$$\mu_{\tau, \beta}^n(\mathbf{dr}, \mathbf{dp}) = \prod_{x \in \mathbb{T}_n} \exp(-\beta(\mathcal{E}_x - \tau r_x) - \mathcal{G}_{\tau, \beta}) \, dr_x dp_x, \quad \beta > 0, \tau \in \mathbb{R}.$$

Herein-above,

$$\mathcal{G}_{\tau, \beta} = \log \left[\sqrt{2\pi\beta^{-1}} \int_{\mathbb{R}} e^{-\frac{\beta}{2}(r^2 - 2\tau r)} \, dr \right] = \log \left[2\pi\beta^{-1} \exp\left(\frac{\tau^2\beta}{2}\right) \right].$$

As usual, the parameters $\beta^{-1} > 0$ and $\tau \in \mathbb{R}$ are called respectively the *temperature* and *tension*. Observe that the function

$$r(\tau, \beta) = \beta^{-1} \partial_{\tau} \mathcal{G}_{\tau, \beta} = \tau \tag{2.2}$$

gives the average equilibrium length in function of the tension τ , and

$$\mathcal{E}(\tau, \beta) = \tau r(\tau, \beta) - \partial_{\beta} \mathcal{G}_{\tau, \beta} = \beta^{-1} + \frac{\tau^2}{2} \tag{2.3}$$

is the corresponding thermodynamic internal energy function. Note that the energy $\mathcal{E}(\tau, \beta)$ is composed by a *thermal* energy β^{-1} and a *mechanical* energy $\frac{\tau^2}{2}$.

2.3. Hydrodynamic limits. Let $\mu_n(\mathbf{dr}, \mathbf{dp})$ be an initial Borel probability distribution on Ω_n . We denote by \mathbb{P}_n the law of the process $\{(\mathbf{r}(t), \mathbf{p}(t)) ; t \geq 0\}$ starting from the measure μ_n , and by \mathbb{E}_n its corresponding expectation. We are given initial continuous profiles of tension $\{\tau_0(u) ; u \in \mathbb{T}\}$ and of temperature $\{\beta_0^{-1}(u) > 0 ; u \in \mathbb{T}\}$. The thermodynamic relations (2.2) and (2.3) give the corresponding initial profiles of elongation and energy as

$$r_0(u) := \tau_0(u) \quad \text{and} \quad e_0(u) := \frac{1}{\beta_0(u)} + \frac{\tau_0^2(u)}{2}, \quad u \in \mathbb{T}. \tag{2.4}$$

The initial distributions μ_n are assumed to satisfy the following mean convergence statements:

$$\frac{1}{n} \sum_{x \in \mathbb{T}_n} G\left(\frac{x}{n}\right) \mathbb{E}_n[r_x(0)] \xrightarrow{n \rightarrow +\infty} \int_{\mathbb{T}} G(u) r_0(u) \, du, \quad (2.5)$$

$$\frac{1}{n} \sum_{x \in \mathbb{T}_n} G\left(\frac{x}{n}\right) \mathbb{E}_n[\mathcal{E}_x(0)] \xrightarrow{n \rightarrow +\infty} \int_{\mathbb{T}} G(u) e_0(u) \, du \quad (2.6)$$

for any test function G that belongs to the set $\mathcal{C}^\infty(\mathbb{T})$ of smooth functions on the torus. We expect the same convergence to happen at the macroscopic time t :

$$\begin{aligned} \frac{1}{n} \sum_{x \in \mathbb{T}_n} G\left(\frac{x}{n}\right) \mathbb{E}_n[r_x(t)] &\xrightarrow{n \rightarrow +\infty} \int_{\mathbb{T}} G(u) r(t, u) \, du, \\ \frac{1}{n} \sum_{x \in \mathbb{T}_n} G\left(\frac{x}{n}\right) \mathbb{E}_n[\mathcal{E}_x(t)] &\xrightarrow{n \rightarrow +\infty} \int_{\mathbb{T}} G(u) e(t, u) \, du, \end{aligned} \quad (2.7)$$

where the macroscopic evolution for the volume and energy profiles follows the system of equations:

$$\begin{cases} \partial_t r(t, u) = \frac{1}{2\gamma} \partial_{uu}^2 r(t, u), \\ \partial_t e(t, u) = \frac{1}{4\gamma} \partial_{uu}^2 \left(e + \frac{r^2}{2} \right)(t, u), \end{cases} \quad (t, u) \in \mathbb{R}_+ \times \mathbb{T}, \quad (2.8)$$

with the initial condition

$$r(0, u) = r_0(u), \quad e(0, u) = e_0(u).$$

The solutions $e(t, \cdot)$, $r(t, \cdot)$ of (2.8) are smooth when $t > 0$ (the system of partial differential equations is parabolic). Note that the evolution of $r(t, u)$ is autonomous of $e(t, u)$. The precise conditions that are needed for the convergence (2.7) are stated in Theorems 3.7 and 3.8 below.

3. MAIN RESULTS

3.1. Notations.

3.1.1. *Discrete Fourier transform.* Let us denote by \hat{f} the Fourier transform of a finite sequence $\{f_x\}_{x \in \mathbb{T}_n}$ of numbers in \mathbb{C} , defined as follows:

$$\hat{f}(k) = \sum_{x \in \mathbb{T}_n} f_x e^{-2i\pi x k}, \quad k \in \hat{\mathbb{T}}_n := \left\{0, \frac{1}{n}, \dots, \frac{n-1}{n}\right\}. \quad (3.1)$$

Reciprocally, for any $f : \hat{\mathbb{T}}_n \rightarrow \mathbb{C}$, we denote by \check{f}_x its inverse Fourier transform given by

$$\check{f}_x = \frac{1}{n} \sum_{k \in \hat{\mathbb{T}}_n} e^{2i\pi x k} f(k), \quad x \in \mathbb{T}_n. \quad (3.2)$$

The Parseval identity reads

$$\|f\|_{\mathbf{L}^2}^2 := \frac{1}{n} \sum_{k \in \hat{\mathbb{T}}_n} |\hat{f}(k)|^2 = \sum_{x \in \mathbb{T}_n} |f_x|^2. \quad (3.3)$$

If $\{f_x\}_{x \in \mathbb{T}_n}$ and $\{g_x\}_{x \in \mathbb{T}_n}$ are two sequences indexed by the discrete torus, their convolution is given by

$$(f * g)_x := \sum_{y \in \mathbb{T}_n} f_y g_{x-y}, \quad x \in \mathbb{T}_n.$$

3.1.2. Continuous Fourier transform. Let $\mathcal{C}(\mathbb{T})$ be the space of continuous, complex valued functions on \mathbb{T} . For any function $G \in \mathcal{C}(\mathbb{T})$, let $\mathcal{F}G : \mathbb{Z} \rightarrow \mathbb{C}$ denote its Fourier transform given as follows:

$$\mathcal{F}G(\eta) := \int_{\mathbb{T}} G(u) e^{-2i\pi u \eta} du, \quad \eta \in \mathbb{Z}. \quad (3.4)$$

Similar identities to (3.2) and (3.3) can easily be written: for instance, we shall repeatedly use the following

$$G(u) = \sum_{\eta \in \mathbb{Z}} \mathcal{F}G(\eta) e^{2i\pi \eta u}, \quad u \in \mathbb{T}. \quad (3.5)$$

Note that when G is smooth the Fourier coefficients satisfy

$$\sup_{\eta \in \mathbb{Z}} \left\{ (1 + \eta^2)^p |\mathcal{F}G(\eta)| \right\} < +\infty, \quad \text{for any } p \in \mathbb{N}. \quad (3.6)$$

If $J : \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{C}$ is defined on a two-dimensional torus, we still denote by $\mathcal{F}J(\eta, v)$, $(\eta, v) \in \mathbb{Z} \times \mathbb{T}$, its Fourier transform with respect to the first variable. We equip the set $\mathcal{C}^\infty(\mathbb{T} \times \mathbb{T})$ of smooth (with respect to the first variable) functions with the norm

$$\|J\|_0 := \sum_{\eta \in \mathbb{Z}} \sup_{v \in \mathbb{T}} |\mathcal{F}J(\eta, v)|. \quad (3.7)$$

Let \mathcal{A}_0 be the completion of $\mathcal{C}^\infty(\mathbb{T} \times \mathbb{T})$ in this norm and $(\mathcal{A}'_0, \|\cdot\|'_0)$ its dual space.

3.1.3. A fundamental example. In what follows, we often consider the discrete Fourier transform associated to a function $G \in \mathcal{C}(\mathbb{T})$, and to avoid any confusion we introduce a new notation: let $\mathcal{F}_n G : \hat{\mathbb{T}}_n \rightarrow \mathbb{C}$ be the discrete Fourier transform of the finite sequence $\{G(\frac{x}{n})\}_{x \in \mathbb{T}_n}$ defined similarly to (3.1) as

$$\mathcal{F}_n G(k) := \sum_{x \in \mathbb{T}_n} G\left(\frac{x}{n}\right) e^{-2i\pi x k}, \quad k \in \hat{\mathbb{T}}_n.$$

In particular, we have the Parseval identity

$$\sum_{x \in \mathbb{T}_n} G\left(\frac{x}{n}\right) f_x^\star = \frac{1}{n} \sum_{k \in \hat{\mathbb{T}}_n} (\mathcal{F}_n G)(k) \hat{f}^\star(k). \quad (3.8)$$

Furthermore, note that

$$\frac{1}{n} \mathcal{F}_n G \left(\frac{\eta}{n} \right) \xrightarrow{n \rightarrow \infty} \mathcal{F} G(\eta), \quad \text{for any } \eta \in \mathbb{Z}.$$

3.2. Assumptions on initial data. Without losing too much of generality, one can put natural assumptions on the initial probability measure $\mu_n(\mathrm{d}\mathbf{r}, \mathrm{d}\mathbf{p})$.

The first assumption concerns the mean of the initial configurations, and is sufficient in order to derive the first of the hydrodynamic equations (2.8) :

Assumption 3.1. • *The initial total energy can be random but with uniformly bounded expectation:*

$$\sup_{n \geq 1} \mathbb{E}_n \left[\frac{1}{n} \sum_{x \in \mathbb{T}_n} \mathcal{E}_x(0) \right] < +\infty. \quad (3.9)$$

- *We assume that there exist continuous initial profiles $r_0 : \mathbb{T} \rightarrow \mathbb{R}$ and $e_0 : \mathbb{T} \rightarrow (0, +\infty)$ such that*

$$\mathbb{E}_n[p_x(0)] = 0, \quad \mathbb{E}_n[r_x(0)] = r_0\left(\frac{x}{n}\right) \quad \text{for any } x \in \mathbb{T}_n \quad (3.10)$$

and for any $G \in \mathcal{C}^\infty(\mathbb{T})$

$$\frac{1}{n} \sum_{x \in \mathbb{T}_n} G\left(\frac{x}{n}\right) \mathbb{E}_n[\mathcal{E}_x(0)] \xrightarrow{n \rightarrow +\infty} \int_{\mathbb{T}} G(u) e_0(u) \, \mathrm{d}u. \quad (3.11)$$

Identity (3.10), in particular, implies the mean convergence of the initial elongation:

$$\frac{1}{n} \sum_{x \in \mathbb{T}_n} G\left(\frac{x}{n}\right) \mathbb{E}_n[r_x(0)] \xrightarrow{n \rightarrow +\infty} \int_{\mathbb{T}} G(u) r_0(u) \, \mathrm{d}u, \quad (3.12)$$

for any $G \in \mathcal{C}^\infty(\mathbb{T})$.

Remark 3.2. By energy conservation (3.9) implies that

$$\sup_{n \geq 1} \mathbb{E}_n \left[\frac{1}{n} \sum_{x \in \mathbb{T}_n} \mathcal{E}_x(t) \right] < +\infty, \quad \text{for all } t \geq 0. \quad (3.13)$$

Remark 3.3. Conditions in (3.10) are assumed in order to simplify the proof, but they can be easily relaxed.

Next assumption is important to obtain the macroscopic equation for the energy in (2.8). It concerns the energy spectrum of fluctuations around the means at initial time. Define the *initial thermal energy spectrum* $\mathbf{u}_n(0, k)$, $k \in \widehat{\mathbb{T}}_n$, as follows: let $\widehat{r}(0, k)$ and $\widehat{p}(0, k)$ denote respectively the Fourier transforms of the initial random configurations $\{r_x(0)\}_{x \in \mathbb{T}_n}$ and $\{p_x(0)\}_{x \in \mathbb{T}_n}$, and let

$$\mathbf{u}_n(0, k) := \frac{1}{2n} \mathbb{E}_n \left[|\widehat{p}(0, k)|^2 + |\widehat{r}(0, k) - \mathbb{E}_n[\widehat{r}(0, k)]|^2 \right], \quad k \in \widehat{\mathbb{T}}_n. \quad (3.14)$$

Due to the Parseval identity (3.3) we have

$$\frac{1}{n} \sum_{k \in \widehat{\mathbb{T}}_n} \mathbf{u}_n(0, k) = \frac{1}{2n} \sum_{x \in \mathbb{T}_n} \mathbb{E}_n \left[p_x^2(0) + (r_x(0) - \mathbb{E}_n[r_x(0)])^2 \right].$$

Assumption 3.4. (Square integrable initial thermal energy spectrum)

$$\sup_{n \geq 1} \left\{ \frac{1}{n} \sum_{k \in \widehat{\mathbb{T}}_n} \mathbf{u}_n^2(0, k) \right\} < +\infty. \quad (3.15)$$

Remark 3.5. Assumptions 3.1 and 3.4 are satisfied if the measures μ_n are given by local Gibbs measures (non homogeneous product), corresponding to the given initial profiles of tension and temperature $\{\tau_0(u), \beta_0^{-1}(u) ; u \in \mathbb{T}\}$, defined as follows:

$$d\mu_{\tau_0(\cdot), \beta_0(\cdot)}^n = \prod_{x \in \mathbb{T}_n} \exp \left\{ -\beta_0\left(\frac{x}{n}\right) \left(\mathcal{E}_x - \tau_0\left(\frac{x}{n}\right) r_x \right) - \mathcal{G}_{\tau_0(\frac{x}{n}), \beta_0(\frac{x}{n})} \right\} dr_x dp_x.$$

with $r_0(u) = \tau_0(u)$ and $e_0(u) = \beta_0^{-1}(u) + \frac{r_0^2(u)}{2}$, see Sections 9.2.3-2.5 in [6]. Notice that our assumptions are much more general, as we do not assume any specific condition on the correlation structure of μ_n . In particular microcanonical versions of (3.5), where total energy and total volumes are conditioned at fixed values ne and nr , are included by our assumptions.

Remark 3.6. Intuitively, our assumptions state that the initial energy has a mechanical part, related to $\tau_0(\cdot)$ that concentrates on the longest wavelength (i.e. around $k = 0$) while the thermal energy has a square integrable density in k .

3.3. Formulation of mean convergence. In this section we state two theorems dealing with the mean convergence of the two conserved quantities, namely the elongation and energy. The first one (Theorem 3.7) is proved straightforwardly in Section 3.4 below. The second one is more involved, and is the main subject of the present paper.

Theorem 3.7 (Mean convergence of the elongation profile). *Assume that $\{\mu_n\}_{n \in \mathbb{N}}$ is a sequence of probability measures on Ω_n such that (3.10) is satisfied, with $r_0 \in \mathcal{C}(\mathbb{T})$. Let $r(t, u)$ be the solution defined on $\mathbb{R}_+ \times \mathbb{T}$ of the linear diffusive equation:*

$$\begin{cases} \partial_t r(t, u) = \frac{1}{2\gamma} \partial_{uu}^2 r(t, u), & (t, u) \in \mathbb{R}_+ \times \mathbb{T}, \\ r(0, u) = r_0(u). \end{cases} \quad (3.16)$$

Then, for any $G \in \mathcal{C}^\infty(\mathbb{T})$ and $t \in \mathbb{R}_+$,

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{x \in \mathbb{T}_n} G\left(\frac{x}{n}\right) \mathbb{E}_n[p_x(t)] = 0, \quad (3.17)$$

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{x \in \mathbb{T}_n} G\left(\frac{x}{n}\right) \mathbb{E}_n[r_x(t)] = \int_{\mathbb{T}} G(u) r(t, u) du. \quad (3.18)$$

Theorem 3.8 (Mean convergence of the empirical profile of energy). *Let $\{\mu_n\}_{n \in \mathbb{N}}$ be a sequence of probability measures on Ω_n such that Assumptions 3.1 and 3.4 are satisfied. Then, for any smooth function $G : \mathbb{R}_+ \times \mathbb{T} \rightarrow \mathbb{R}$ compactly supported with respect to the time variable $t \in \mathbb{R}_+$, we have*

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{x \in \mathbb{T}_n} \int_{\mathbb{R}_+} G\left(t, \frac{x}{n}\right) \mathbb{E}_n[\mathcal{E}_x(t)] dt = \int_{\mathbb{R}_+ \times \mathbb{T}} G(t, u) e(t, u) dt du, \quad (3.19)$$

where $e(t, u) = e_{\text{mech}}(t, u) + e_{\text{thm}}(t, u)$, with

- the mechanical energy, given by $e_{\text{mech}}(t, u) := \frac{1}{2} (r(t, u))^2$ and the function $r(t, u)$ being the solution of (3.16),
- the thermal energy $e_{\text{thm}}(t, u)$, defined as the solution to

$$\begin{cases} \partial_t e_{\text{thm}}(t, u) = \frac{1}{4\gamma} \partial_{uu}^2 e_{\text{thm}}(t, u) + \frac{1}{2\gamma} (\partial_u r(t, u))^2, \\ e_{\text{thm}}(0, u) = \beta_0^{-1}(u) = e_0(u) - e_{\text{mech}}(0, u) > 0. \end{cases} \quad (3.20)$$

The proof of Theorem 3.8 is presented in Sections 4 – 6.

Remark 3.9. Note that (3.16) and (3.20) are equivalent to the system (2.8). This new way of seeing the macroscopic equations is more convenient, as it naturally arises from the proof. More precisely, using (3.16) we conclude that the mechanical energy $e_{\text{mech}}(t, u)$ satisfies the equation

$$\partial_t e_{\text{mech}}(t, u) = \frac{1}{2\gamma} \left(\partial_{uu}^2 e_{\text{mech}}(t, u) - (\partial_u r(t, u))^2 \right)$$

and the macroscopic energy density function satisfies

$$\begin{cases} \partial_t e(t, u) = \frac{1}{4\gamma} \partial_{uu}^2 (e(t, u) + e_{\text{mech}}(t, u)), \\ e(0, u) = e_0(u). \end{cases}$$

3.4. Proof of the hydrodynamic limit of the elongation. Here we give a simple proof of Theorem 3.7. From the evolution equations (2.1) we have the following identities:

$$\frac{1}{n} \sum_{x \in \mathbb{T}_n} G\left(\frac{x}{n}\right) \mathbb{E}_n[r_x(t) - r_x(0)] = n^2 \int_0^t \frac{1}{n} \sum_{x \in \mathbb{T}_n} G\left(\frac{x}{n}\right) \mathbb{E}_n[p_x(s) - p_{x-1}(s)] ds$$

and

$$\begin{aligned} 2\gamma n^2 \int_0^t \frac{1}{n} \sum_{x \in \mathbb{T}_n} G\left(\frac{x}{n}\right) \mathbb{E}_n p_x(s) ds &= n^2 \int_0^t \frac{1}{n} \sum_{x \in \mathbb{T}_n} G\left(\frac{x}{n}\right) \mathbb{E}_n [r_{x+1}(s) - r_{x-1}(s)] ds \\ &\quad + \frac{1}{n} \sum_{x \in \mathbb{T}_n} G\left(\frac{x}{n}\right) \mathbb{E}_n [p_x(0) - p_x(t)]. \end{aligned}$$

Substituting from the second equation into the first one we conclude that

$$\begin{aligned} \frac{1}{n} \sum_{x \in \mathbb{T}_n} G\left(\frac{x}{n}\right) \mathbb{E}_n[r_x(t) - r_x(0)] &= \int_0^t \frac{1}{2\gamma n} \sum_{x \in \mathbb{T}_n} \Delta_n G\left(\frac{x}{n}\right) \mathbb{E}_n[r_x(s)] ds \\ &\quad + \frac{1}{2\gamma n^2} \sum_{x \in \mathbb{T}_n} \nabla_n G\left(\frac{x}{n}\right) \mathbb{E}_n[p_x(0) - p_x(t)], \end{aligned}$$

where

$$\nabla_n G\left(\frac{x}{n}\right) = n \left(G\left(\frac{x+1}{n}\right) - G\left(\frac{x}{n}\right) \right) \text{ and } \Delta_n G\left(\frac{x}{n}\right) = n \left(\nabla_n G\left(\frac{x}{n}\right) - \nabla_n G\left(\frac{x-1}{n}\right) \right).$$

By energy conservation and Assumption 3.1 it is easy to see that

$$\begin{aligned} \left| \frac{1}{n^2} \sum_{x \in \mathbb{T}_n} \nabla_n G\left(\frac{x}{n}\right) \mathbb{E}_n[p_x(t)] \right|^2 &\leq \frac{1}{n^2} \left(\frac{1}{n} \sum_{x \in \mathbb{T}_n} \left[\nabla_n G\left(\frac{x}{n}\right) \right]^2 \right) \left(\frac{1}{n} \sum_{x \in \mathbb{T}_n} \mathbb{E}_n[p_x^2(t)] \right) \\ &\leq \frac{C(G)}{n} \left\{ \frac{1}{n} \sum_{x \in \mathbb{T}_n} \mathbb{E}_n[\mathcal{E}_x^2(0)] \right\} \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

The above consideration also shows that the sequence

$$\bar{r}^{(n)}(t, u) = \mathbb{E}_n[r_x(t)], \quad u \in \left[\frac{x}{n}, \frac{x+1}{n} \right), \quad n \geq 1,$$

is sequentially compact in the space $\mathcal{C}([0, T], \mathbf{L}_w^2(\mathbb{T}))$ for any $T > 0$. Here $\mathbf{L}_w^2(\mathbb{T})$ denotes the space of square integrable functions on the torus \mathbb{T} equipped with the weak \mathbf{L}^2 topology. Consequently, any limiting point of the sequence satisfies the partial differential equation (3.16) in a weak sense in the class of $\mathbf{L}^2(\mathbb{T})$ functions. Uniqueness of the weak solution of the heat equation gives the convergence claimed in (3.18) and the identification of the limit as the strong solution of (3.16).

Concerning (3.17), from (2.1) we have

$$\begin{aligned} \frac{1}{n} \sum_{x \in \mathbb{T}_n} G\left(\frac{x}{n}\right) \mathbb{E}_n[p_x(t)] &= \frac{e^{-2\gamma n^2 t}}{n} \sum_{x \in \mathbb{T}_n} G\left(\frac{x}{n}\right) \mathbb{E}_n[p_x(0)] \\ &\quad + \int_0^t e^{-2\gamma n^2(t-s)} \sum_{x \in \mathbb{T}_n} \nabla_n^* G\left(\frac{x}{n}\right) \mathbb{E}_n[r_x(s)] ds, \end{aligned}$$

where $\nabla_n^* G\left(\frac{x}{n}\right) = n \left(G\left(\frac{x}{n}\right) - G\left(\frac{x-1}{n}\right) \right)$. Using again energy conservation and the Cauchy-Schwarz inequality, it is easy to see that the right-hand side of the above vanishes as $n \rightarrow \infty$.

Remark 3.10. *Note that we have not used the fact that the initial average of the velocities vanishes. It is possible by standard methods to obtain the results in Theorem 3.7 in probability, but we shall not pursue this here.*

4. TIME-DEPENDENT WIGNER DISTRIBUTIONS

4.1. Wave function for the system of oscillators. Let $\hat{p}(t, k)$ and $\hat{r}(t, k)$, $k \in \hat{\mathbb{T}}_n$, denote, respectively, the Fourier transforms of the momentum and elongation components of the microscopic configuration $\{p_x(t)\}_{x \in \mathbb{T}_n}$ and $\{r_x(t)\}_{x \in \mathbb{T}_n}$, as in (3.1). Since they are real-valued we have, for any $k \in \hat{\mathbb{T}}_n$,

$$\hat{p}^*(t, k) = \sum_{x \in \mathbb{T}_n} e^{2\pi i k x} p_x(t) = \hat{p}(t, -k) \quad \text{and likewise} \quad \hat{r}^*(t, k) = \hat{r}(t, -k). \quad (4.1)$$

The *wave function* associated to the dynamics is defined as

$$\psi_x(t) := r_x(t) + i p_x(t), \quad x \in \mathbb{T}_n,$$

and its Fourier transform equals

$$\hat{\psi}(t, k) := \hat{r}(t, k) + i \hat{p}(t, k), \quad k \in \hat{\mathbb{T}}_n.$$

Taking into account (4.1) we obtain

$$\begin{aligned} \hat{p}(t, k) &= \frac{1}{2i} (\hat{\psi}(t, k) - \hat{\psi}^*(t, -k)), \\ \hat{r}(t, k) &= \frac{1}{2} (\hat{\psi}(t, k) + \hat{\psi}^*(t, -k)). \end{aligned}$$

With these definitions we have $|\psi_x|^2 = 2\mathcal{E}_x$ and the initial thermal energy spectrum, defined in (3.14), can be recovered as

$$\mathfrak{u}_n(0, k) = \frac{1}{2n} \mathbb{E}_n \left[|\hat{\psi}(0, k) - \mathbb{E}_n[\hat{\psi}(0, k)]|^2 \right]. \quad (4.2)$$

After a straightforward calculation, the equation that governs the time evolution of the wave function can be deduced from (2.1) as follows:

$$\begin{aligned} d\hat{\psi}(t, k) &= -n^2 \left(2i \sin^2(\pi k) \hat{\psi}(t, k) + \sin(2\pi k) \hat{\psi}^*(t, -k) \right) dt \\ &\quad - \frac{1}{n} \sum_{k' \in \hat{\mathbb{T}}_n} \left\{ \hat{\psi}(t^-, k - k') - \hat{\psi}^*(t^-, k' - k) \right\} d\hat{\mathcal{N}}(t, k'), \end{aligned} \quad (4.3)$$

with initial condition $\hat{\psi}(0, k) = \hat{r}(0, k)$. The semi-martingales $\{\hat{\mathcal{N}}(t, k) ; t \geq 0\}$ are defined as

$$\hat{\mathcal{N}}(t, k) := \sum_{x \in \mathbb{T}_n} \mathcal{N}_x(\gamma n^2 t) e^{-2i\pi x k}, \quad k \in \hat{\mathbb{T}}_n.$$

Observe that we have $\hat{\mathcal{N}}^*(t, k) = \hat{\mathcal{N}}(t, -k)$. In addition, its mean and covariance equal respectively

$$\begin{aligned} \langle d\hat{\mathcal{N}}(t, k) \rangle &= \gamma n^3 \delta_{k,0} dt, \\ \langle d\hat{\mathcal{N}}(t, k), d\hat{\mathcal{N}}(t, k') \rangle &= \gamma n^3 t \delta_{k, -k'} dt, \end{aligned}$$

where $\delta_{x,y}$ is the usual Kronecker delta function, which equals 1 if $x = y$ and 0 otherwise. The conservation of energy, and Parseval's identity, imply together that:

$$\|\widehat{\psi}(t)\|_{\mathbf{L}^2} = \|\widehat{\psi}(0)\|_{\mathbf{L}^2} \quad \text{for all } t \geq 0. \quad (4.4)$$

4.2. Wigner distributions and Fourier transforms. The *Wigner distribution* $\mathbf{W}_n^+(t)$ corresponding to the wave function $\psi(t)$ is a distribution defined as

$$\langle \mathbf{W}_n^+(t), G \rangle := \frac{1}{n} \sum_{k \in \widehat{\mathbb{T}}_n} \sum_{\eta \in \mathbb{Z}} W_n^+(t, \eta, k) (\mathcal{F}G)^*(\eta, k), \quad G \in \mathcal{C}^\infty(\mathbb{T} \times \mathbb{T}), \quad (4.5)$$

where the *Wigner function* $W_n^+(t)$ is given for any $(k, \eta) \in \widehat{\mathbb{T}}_n \times \mathbb{Z}$ by

$$W_n^+(t, \eta, k) := \frac{1}{2n} \mathbb{E}_n \left[\widehat{\psi} \left(t, k + \frac{\eta}{n} \right) \widehat{\psi}^*(t, k) \right]. \quad (4.6)$$

Here, we use the mapping $\mathbb{Z} \ni \eta \mapsto \frac{\eta}{n} \in \widehat{\mathbb{T}}_n$, and $\mathcal{F}G(\eta, v)$ denotes the Fourier transform with respect to the first variable. The main interest of the Wigner distribution is that mean convergence of the empirical energy profile (3.19) can be restated in terms of convergence of Wigner functions (more precisely, their Laplace transforms, see Theorem 6.5 below), thanks to the following identity: if $G(u, v) \equiv G(u)$ does not depend on the second variable $v \in \mathbb{T}$, then

$$\langle \mathbf{W}_n^+(t), G \rangle = \frac{1}{2n} \sum_{x \in \mathbb{T}_n} \mathbb{E}_n[|\psi_x(t)|^2] G^*\left(\frac{x}{n}\right) = \frac{1}{n} \sum_{x \in \mathbb{T}_n} \mathbb{E}_n[\mathcal{E}_x(t)] G^*\left(\frac{x}{n}\right). \quad (4.7)$$

In other words the k -average of the Wigner distribution gives the Fourier transform of the energy:

$$\frac{1}{n} \sum_{k \in \widehat{\mathbb{T}}_n} W_n^+(t, \eta, k) = \frac{1}{2n} \sum_{x \in \mathbb{T}_n} e^{-2\pi i x \frac{\eta}{n}} \mathbb{E}_n[|\psi_x(t)|^2] = \frac{1}{n} \mathbb{E}_n \left[\widehat{\mathcal{E}} \left(t, \frac{\eta}{n} \right) \right], \quad \eta \in \mathbb{Z}. \quad (4.8)$$

To close the equations governing the evolution of $\mathbf{W}_n^+(t)$, we need to define three other Wigner-type functions. We let

$$W_n^-(t, \eta, k) := \frac{1}{2n} \mathbb{E}_n \left[\widehat{\psi}^* \left(t, -k - \frac{\eta}{n} \right) \widehat{\psi}(t, -k) \right] = (W_n^+)^*(t, -\eta, -k), \quad (4.9)$$

$$Y_n^+(t, \eta, k) := \frac{1}{2n} \mathbb{E}_n \left[\widehat{\psi} \left(t, k + \frac{\eta}{n} \right) \widehat{\psi}(t, -k) \right], \quad (4.10)$$

$$Y_n^-(t, \eta, k) := \frac{1}{2n} \mathbb{E}_n \left[\widehat{\psi}^* \left(t, -k - \frac{\eta}{n} \right) \widehat{\psi}^*(t, k) \right] = (Y_n^+)^*(t, -\eta, -k). \quad (4.11)$$

4.3. Properties of the Wigner distributions.

4.3.1. *Weak convergence.* From (3.13) we have that, for any $n \geq 1$ and $G \in \mathcal{C}^\infty(\mathbb{T} \times \mathbb{T})$,

$$\begin{aligned} & |\langle \mathbf{W}_n^+(t), G \rangle| \\ & \leq \frac{1}{(2n)^2} \sup_{\eta \in \mathbb{Z}} \left\{ \sum_{k \in \hat{\mathbb{T}}_n} \mathbb{E}_n \left[\left| \hat{\psi} \left(t, k + \frac{\eta}{n} \right) \right|^2 \right] \right\}^{1/2} \left\{ \sum_{k \in \hat{\mathbb{T}}_n} \mathbb{E}_n \left[\left| \hat{\psi}(t, k) \right|^2 \right] \right\}^{1/2} \|G\|_0 \\ & \leq \frac{\|G\|_0}{2n} \mathbb{E}_n \left[\sum_{x \in \mathbb{T}_n} \mathcal{E}_x(t) \right] \leq C \|G\|_0, \end{aligned} \quad (4.12)$$

where the norm $\|G\|_0$ is defined by (3.7). Hence, for the corresponding dual norm, we have the bound

$$\sup_{t \geq 0} \sup_{n \geq 1} \left\{ \|\mathbf{W}_n^+(t)\|'_0 + \|\mathbf{W}_n^-(t)\|'_0 + \|\mathbf{Y}_n^+(t)\|'_0 + \|\mathbf{Y}_n^-(t)\|'_0 \right\} \leq 4C. \quad (4.13)$$

Condition (3.11) ensures that, if $G(u, v) \equiv G(u)$ at the initial time $t = 0$ we have

$$\lim_{n \rightarrow +\infty} \langle \mathbf{W}_n^\pm(0), G \rangle = \int_{\mathbb{T}} e_0(u) G^*(u) du,$$

4.3.2. *Decomposition into the mechanical and fluctuating part.* We decompose the wave function into its mean and its fluctuation:

$$\psi_x(t) = \bar{\psi}_x(t) + \tilde{\psi}_x(t), \quad x \in \mathbb{T}_n, \quad t \geq 0.$$

Notice that for the initial data we have $\bar{\psi}_x(0) = r_0\left(\frac{x}{n}\right)$. It need not be true for $t > 0$. The Fourier transform of the sequences $\{\bar{\psi}_x(t)\}$ and $\{\tilde{\psi}_x(t)\}$ shall be denoted by $\hat{\bar{\psi}}(t, k)$ and $\hat{\tilde{\psi}}(t, k)$. The deterministic function $\hat{\bar{\psi}}(t, k)$ satisfies the autonomous equation:

$$\begin{aligned} d\hat{\bar{\psi}}(t, k) &= -n^2 \left(2i \sin^2(\pi k) \hat{\bar{\psi}}(t, k) + \sin(2\pi k) \hat{\bar{\psi}}^*(t, -k) \right) dt \\ &\quad - n^2 \gamma \left\{ \hat{\bar{\psi}}(t, k) - \hat{\bar{\psi}}^*(t, -k) \right\} dt, \end{aligned} \quad (4.14)$$

with initial condition $\hat{\bar{\psi}}(0, k) = (\mathcal{F}_n r_0)(k)$.

The Wigner distribution $\mathbf{W}_n^+(t)$ can be decomposed accordingly as follows:

$$\mathbf{W}_n^+(t) = \overline{\mathbf{W}}_n^+(t) + \widetilde{\mathbf{W}}_n^+(t), \quad (4.15)$$

where the Fourier transforms of $\overline{\mathbf{W}}_n^+(t)$ and $\widetilde{\mathbf{W}}_n^+(t)$ are given by

$$\overline{W}_n^+(t, \eta, k) := \frac{1}{2n} \hat{\bar{\psi}} \left(t, k + \frac{\eta}{n} \right) \hat{\bar{\psi}}^*(t, k) \quad (4.16)$$

$$\widetilde{W}_n^+(t, \eta, k) := \frac{1}{2n} \mathbb{E}_n \left[\hat{\tilde{\psi}} \left(t, k + \frac{\eta}{n} \right) \hat{\tilde{\psi}}^*(t, k) \right], \quad (4.17)$$

for any $(\eta, k) \in \mathbb{Z} \times \hat{\mathbb{T}}_n$.

At initial time $t = 0$ we write $\overline{W}_n^+(\eta, k) := \overline{W}_n^+(0, \eta, k)$ and $\widetilde{W}_n^+(\eta, k) := \widetilde{W}_n^+(0, \eta, k)$. We have

$$\overline{W}_n^+(\eta, k) = \frac{1}{2n}(\mathcal{F}_n r_0) \left(k + \frac{\eta}{n} \right) (\mathcal{F}_n r_0)^*(k), \quad (4.18)$$

Recalling (4.2) we write

$$\mathbf{u}_n(0, k) = \frac{1}{2n} \mathbb{E}_n \left[|\widehat{\psi}(0, k)|^2 \right].$$

Following (4.10) one can easily write similar definitions for $\overline{\mathbf{W}}_n^-$, $\widetilde{\mathbf{W}}_n^-$ and the respective $\overline{\mathbf{Y}}_n^\pm$, $\widetilde{\mathbf{Y}}_n^\pm$ distributions. Due to the fact that $\bar{\psi}_x(0) = r_0(\frac{x}{n})$ and is real-valued (and extending the convention of omitting the argument $t = 0$ for all Wigner-type distributions) we have

$$\overline{W}_n^-(\eta, k) = \overline{W}_n^+(\eta, k) = \overline{Y}_n^+(\eta, k) = \overline{Y}_n^-(\eta, k) = W_n(r_0; \eta, k), \quad (4.19)$$

where we define

$$W_n(r; \eta, k) := \frac{1}{2n}(\mathcal{F}_n r) \left(k + \frac{\eta}{n} \right) (\mathcal{F}_n r)^*(k), \quad (4.20)$$

for any $r \in \mathcal{C}(\mathbb{T})$ and $(\eta, k) \in \mathbb{Z} \times \widehat{\mathbb{T}}_n$.

4.3.3. *Asymptotic of \widetilde{W}_n^+ .* Throughout the remainder of the paper we shall use the following notation: given a function $f : \widehat{\mathbb{T}}_n \rightarrow \mathbb{C}$ we denote its k -average by

$$[f(\cdot)]_n := \frac{1}{n} \sum_{k \in \widehat{\mathbb{T}}_n} f(k). \quad (4.21)$$

The initial fluctuating Wigner function is related, as $n \rightarrow +\infty$, to the initial thermal energy $e_{\text{thm}}(0, u) = e_0(u) - r_0^2(u)/2$ as follows:

$$\left[\widetilde{W}_n^+(\eta, \cdot) \right]_n \xrightarrow[n \rightarrow \infty]{} (\mathcal{F} e_{\text{thm}}(0, \cdot))(\eta), \quad \text{for any } \eta \in \mathbb{Z}. \quad (4.22)$$

The last convergence follows from Assumption 3.1 and from an explicit computation that yields:

$$\left[\widetilde{W}_n^+(\eta, \cdot) \right]_n = \frac{1}{n} \left\{ \sum_{x \in \mathbb{T}_n} \mathbb{E}_n[\mathcal{E}_x(0)] e^{-2\pi i \eta \frac{x}{n}} - \mathcal{F}_n \left(\frac{r_0^2}{2} \right) \left(\frac{\eta}{n} \right) \right\}.$$

In addition, condition (3.15) implies that

$$w_* := \sup_{n \geq 1} \sup_{\eta \in \mathbb{Z}} \left\{ \sum_{\iota \in \{-, +\}} \left[\left| \widetilde{W}_n^\iota(\eta, \cdot) \right|^2 + \left| \widetilde{Y}_n^\iota(\eta, \cdot) \right|^2 \right]_n \right\} < +\infty. \quad (4.23)$$

4.3.4. *Wigner distribution associated to a macroscopic smooth profile.* Given a continuous real-valued function $\{r := r(u) ; u \in \mathbb{T}\}$, define

$$W(r ; \eta, \xi) := \frac{1}{2}(\mathcal{F}r)(\xi + \eta)(\mathcal{F}r)^*(\xi), \quad (\eta, \xi) \in \mathbb{Z}^2. \quad (4.24)$$

Observe that, for any $\eta \in \mathbb{Z}$,

$$\sum_{\xi \in \mathbb{Z}} W(r ; \eta, \xi) = \frac{1}{2} \mathcal{F}(r^2)(\eta). \quad (4.25)$$

Proposition 4.1. *Suppose that $r \in \mathcal{C}(\mathbb{T})$ and $G \in \mathcal{C}^\infty(\mathbb{T} \times \mathbb{T})$. Then*

$$\begin{aligned} \lim_{n \rightarrow +\infty} \sum_{\eta \in \mathbb{Z}} \sum_{\eta \in \mathbb{Z}} \left[W_n(r ; \eta, \cdot)(\mathcal{F}J)^*(\eta, \cdot) \right]_n &= \sum_{\eta \in \mathbb{Z}} \left(\sum_{\xi \in \mathbb{Z}} W(r ; \eta, \xi) \right) (\mathcal{F}G)^*(\eta, 0) \\ &= \frac{1}{2} \int_{\mathbb{T}} r^2(u) G^*(u, 0) \, du. \end{aligned} \quad (4.26)$$

Proof. We prove the proposition under the assumption that $r \in \mathcal{C}^\infty(\mathbb{T})$. The general case can be obtained by an approximation of a continuous initial profile by a sequence of smooth ones.

Using the dominated convergence theorem we conclude that the expression on the left-hand side of (4.26) equals $\sum_{\eta \in \mathbb{Z}} \lim_{n \rightarrow +\infty} f_n(\eta)$, with

$$f_n(\eta) := \left[\overline{W}_n^+(\eta, \cdot)(\mathcal{F}G)^*(\eta, \cdot) \right]_n, \quad \eta \in \mathbb{Z}. \quad (4.27)$$

This can be written as

$$f_n(\eta) := \frac{1}{2n^2} \sum_{k \in \widehat{\mathbb{T}}_n} \sum_{x, x' \in \mathbb{T}_n} e^{-2\pi i \eta \frac{x}{n}} e^{2\pi i k(x' - x)} r\left(\frac{x}{n}\right) r\left(\frac{x'}{n}\right) (\mathcal{F}G)^*(\eta, k).$$

Using Fourier representation (see (3.5)), we can write

$$f_n(\eta) = \frac{1}{2n^2} \sum_{\xi, \xi' \in \mathbb{Z}} \sum_{m, x, x' \in \mathbb{T}_n} e^{-2\pi i(m + \xi + \eta) \frac{x}{n}} e^{2\pi i(m + \xi') \frac{x'}{n}} (\mathcal{F}r)^*(\xi)(\mathcal{F}r)(\xi') (\mathcal{F}G)^*\left(\eta, \frac{m}{n}\right).$$

Due to smoothness of $r(\cdot)$, its Fourier coefficients decay rapidly. Therefore for a fixed $\eta \in \mathbb{Z}$ and for any $\rho \in (0, 1)$ we have that the limit $\lim_{n \rightarrow +\infty} f_n(\eta)$ equals

$$\lim_{n \rightarrow +\infty} \frac{1}{2n^2} \sum_{\substack{|\xi| \leq n^\rho \\ |\xi'| \leq n^\rho}} \sum_{m, x, x' \in \mathbb{T}_n} e^{-2\pi i(m + \xi + \eta) \frac{x}{n}} e^{2\pi i(m + \xi') \frac{x'}{n}} (\mathcal{F}r)^*(\xi)(\mathcal{F}r)(\xi') (\mathcal{F}G)^*\left(\eta, \frac{m}{n}\right).$$

Summing over x, x' we conclude that

$$\begin{aligned} & \lim_{n \rightarrow +\infty} f_n(\eta) \\ &= \lim_{n \rightarrow +\infty} \frac{1}{2} \sum_{\substack{|\xi| \leq n^\rho \\ |\xi'| \leq n^\rho}} \sum_{m \in \mathbb{T}_n} 1_{\mathbb{Z}}\left(\frac{m + \xi + \eta}{n}\right) 1_{\mathbb{Z}}\left(\frac{m + \xi'}{n}\right) (\mathcal{F}r)^*(\xi)(\mathcal{F}r_0)(\xi')(\mathcal{F}G)^*\left(\eta, \frac{m}{n}\right), \end{aligned}$$

where $1_{\mathbb{Z}}$ is the indicator function of the set of integers. Taking into account the fact that $m \in \mathbb{T}_n$ and the magnitude of ξ, ξ' is negligible when compared with n we conclude that

$$\begin{aligned} \lim_{n \rightarrow +\infty} f_n(\eta) &= \lim_{n \rightarrow +\infty} \frac{1}{2} \sum_{\substack{|\xi| \leq n^\rho \\ |\xi'| \leq n^\rho}} \sum_{m \in \mathbb{T}_n} \delta_{n, m + \xi + \eta} \delta_{n, m + \xi'} (\mathcal{F}r)^*(\xi)(\mathcal{F}r)(\xi')(\mathcal{F}G)^*\left(\eta, \frac{m}{n}\right) \\ &\quad + \lim_{n \rightarrow +\infty} \frac{1}{2} \sum_{\substack{|\xi| \leq n^\rho \\ |\xi'| \leq n^\rho}} \sum_{m \in \mathbb{T}_n} \delta_{0, m + \xi + \eta} \delta_{0, m + \xi'} (\mathcal{F}r)^*(\xi)(\mathcal{F}r)(\xi')(\mathcal{F}G)^*\left(\eta, \frac{m}{n}\right) \\ &= \frac{1}{2} (\mathcal{F}G)^*(\eta, 0) \sum_{\xi \in \mathbb{Z}} (\mathcal{F}r)^*(\xi)(\mathcal{F}r)(\xi + \eta) \end{aligned}$$

and formula (4.26) follows. \square

In particular, for the initial conditions of our dynamics, Proposition 4.1 implies:

$$\begin{aligned} \lim_{n \rightarrow +\infty} \sum_{\eta \in \mathbb{Z}} \left[\overline{W}_n^+(\eta, \cdot)(\mathcal{F}G)^*(\eta, \cdot) \right]_n &= \sum_{\xi, \eta \in \mathbb{Z}} W(r_0; \eta, \xi)(\mathcal{F}G)^*(\eta, 0) \\ &= \frac{1}{2} \int_{\mathbb{T}} r_0^2(u) G^*(u, 0) du. \end{aligned} \tag{4.28}$$

One of the main point of the proof of our theorem is to show that this convergence holds for any macroscopic time $t > 0$, i.e. that for any compactly supported $G \in \mathcal{C}^\infty(\mathbb{R}_+ \times \mathbb{T}^2)$ we have

$$\begin{aligned} \lim_{n \rightarrow +\infty} \sum_{\eta \in \mathbb{Z}} \int_0^{+\infty} \left[\overline{W}_n^+(t, \eta, \cdot)(\mathcal{F}G)^*(t, \eta, \cdot) \right]_n dt &= \sum_{\xi, \eta \in \mathbb{Z}} \int_0^{+\infty} W(r_t; \eta, \xi)(\mathcal{F}G)^*(t, \eta, 0) dt \\ &= \frac{1}{2} \int_{\mathbb{R}_+ \times \mathbb{T}} r^2(t, u) G^*(t, u) dt du. \end{aligned} \tag{4.29}$$

This would amount to showing that the Wigner distribution $\overline{W}_n^+(t)$ associated to $\overline{\psi}_x(t)$ is asymptotically equivalent to the one corresponding to the macroscopic profile $r(t, \cdot)$ via (4.24). This fact is not a consequence of Theorem 3.7, that implies only a weak convergence of $\overline{\psi}_x(t)$ to $r(t, \cdot)$. We will prove (4.29) in

Proposition 6.1 below, showing the convergence of the corresponding Laplace transforms (see (6.2)).

5. STRATEGY OF THE PROOF AND EXPLICIT RESOLUTIONS

5.1. Laplace transform of Wigner functions. Since our subsequent argument is based on an application of the Laplace transform of the Wigner functions, we give some explicit formulas for the object that can be written in case of our model. For any bounded complex-valued, Borel measurable function $\mathbb{R}_+ \ni t \mapsto f_t$ we define the Laplace operator \mathcal{L} as:

$$\mathcal{L}(f)(\lambda) := \int_0^{+\infty} e^{-\lambda t} f_t \, dt, \quad \lambda > 0.$$

Given the solution $r_t = r(t, \cdot)$ of (3.16), we define the Laplace transform of the Wigner distribution as follows: for any $(\lambda, \eta, \xi) \in \mathbb{R}_+ \times \mathbb{Z}^2$,

$$w(r; \eta, \xi)(\lambda) = \mathcal{L}(W(r; \eta, \xi))(\lambda) = \int_0^{+\infty} e^{-\lambda t} W(r_t; \eta, \xi) \, dt.$$

The following formulas are easily deduced, by a direct calculation, from (3.16):

Lemma 5.1. *For any $(\lambda, \eta, \xi) \in \mathbb{R}_+ \times \mathbb{Z}^2$ we have*

$$w(r; \eta, \xi)(\lambda) = \frac{W(r_0; \eta, \xi)}{\frac{2\pi^2}{\gamma} [\xi^2 + (\eta + \xi)^2] + \lambda}. \quad (5.1)$$

Consequently,

$$\begin{aligned} \frac{1}{2} \mathcal{L}(\mathcal{F}((\partial_u r_t)^2)(\eta))(\lambda) &= \frac{1}{2} \int_0^{+\infty} e^{-\lambda t} \mathcal{F}((\partial_u r_t)^2)(\eta) \, dt \\ &= 4\pi^2 \sum_{\xi \in \mathbb{Z}} (\eta + \xi) \xi w(r; \eta, \xi)(\lambda). \end{aligned} \quad (5.2)$$

Finally, we define \mathbf{w}_n^+ the Laplace transform of \mathbf{W}_n^+ as the tempered distribution given for any $G \in \mathcal{C}^\infty(\mathbb{T} \times \mathbb{T})$ by

$$\langle \mathbf{w}_n^+(\lambda), G \rangle = \int_0^{+\infty} e^{-\lambda t} \langle \mathbf{W}_n^+(t), G \rangle \, dt := \frac{1}{n} \sum_{k \in \widehat{\mathbb{T}}_n} \sum_{\eta \in \mathbb{Z}} w_n^+(\lambda, \eta, k) (\mathcal{F}G)^\star(\eta, k), \quad (5.3)$$

for any $\lambda > 0$, where w_n^+ is the Laplace transform of W_n^+ as follows:

$$w_n^+(\lambda, \eta, k) := \int_0^{+\infty} e^{-\lambda t} W_n^+(t, \eta, k) \, dt, \quad (\lambda, \eta, k) \in \mathbb{R}_+ \times \mathbb{Z} \times \widehat{\mathbb{T}}_n.$$

In a similar fashion we can also define $\mathbf{w}_n^-(\lambda)$ and $\mathbf{y}_n^\pm(\lambda)$ the Laplace transforms of $\mathbf{W}_n^-(t)$ and $\mathbf{Y}_n^\pm(t)$, respectively and their counterparts w_n^- , and y_n^\pm .

5.2. Dynamics of the Wigner distributions. Using (4.3), one can first write a closed system of evolution equations for $W_n^\pm(t), Y_n^\pm(t)$ defined respectively in (4.6), (4.9), (4.10), (4.11).

For that purpose we define two functions $\delta_n s$ and $\sigma_n s$ as follows: for any $(\eta, k) \in \mathbb{Z} \times \hat{\mathbb{T}}_n$,

$$\begin{aligned} (\delta_n s)(\eta, k) &:= 2n \left(\sin^2 \left(\pi \left(k + \frac{\eta}{n} \right) \right) - \sin^2(\pi k) \right), \\ (\sigma_n s)(\eta, k) &:= 2 \left(\sin^2 \left(\pi \left(k + \frac{\eta}{n} \right) \right) + \sin^2(\pi k) \right). \end{aligned} \quad (5.4)$$

For the brevity sake, we drop the variables $(t, \eta, k) \in \mathbb{R}_+ \times \mathbb{Z} \times \hat{\mathbb{T}}_n$ from the subsequent notation. From (4.3) we conclude that:

$$\left\{ \begin{aligned} \partial_t W_n^+ &= -in(\delta_n s) W_n^+ - n^2 \sin(2\pi k) Y_n^+ - n^2 \sin \left(2\pi \left(k + \frac{\eta}{n} \right) \right) Y_n^- \\ &\quad + \gamma n^2 \mathbb{L}(2W_n^+ - Y_n^+ - Y_n^-), \\ \partial_t Y_n^+ &= n^2 \sin(2\pi k) W_n^+ - in^2(\sigma_n s) Y_n^+ - n^2 \sin \left(2\pi \left(k + \frac{\eta}{n} \right) \right) W_n^- \\ &\quad + \gamma n^2 \mathbb{L}(2Y_n^+ - W_n^+ - W_n^-) + \gamma n \sum_{k \in \hat{\mathbb{T}}_n} (Y_n^- - Y_n^+), \\ \partial_t Y_n^- &= n^2 \sin \left(2\pi \left(k + \frac{\eta}{n} \right) \right) W_n^+ + in^2(\sigma_n s) Y_n^- - n^2 \sin(2\pi k) W_n^- \\ &\quad + \gamma n^2 \mathbb{L}(2Y_n^- - W_n^+ - W_n^-) + \gamma n \sum_{k \in \hat{\mathbb{T}}_n} (Y_n^+ - Y_n^-), \\ \partial_t W_n^- &= in(\delta_n s) W_n^- + n^2 \sin \left(2\pi \left(k + \frac{\eta}{n} \right) \right) Y_n^+ + n^2 \sin(2\pi k) Y_n^- \\ &\quad + \gamma n^2 \mathbb{L}(2W_n^- - Y_n^+ - Y_n^-), \end{aligned} \right. \quad (5.5)$$

where \mathbb{L} is the operator that is defined for any $f : \mathbb{Z} \times \hat{\mathbb{T}}_n \rightarrow \mathbb{C}$ as

$$(\mathbb{L}f)(\eta, k) := [f(\eta, \cdot)]_n - f(\eta, k), \quad (\eta, k) \in \mathbb{Z} \times \hat{\mathbb{T}}_n.$$

Recalling the decomposition (4.15) and the evolution equations (4.14), we have that $\overline{W}_n^\pm(t), \overline{Y}_n^\pm(t)$ satisfy the autonomous equations:

$$\left\{ \begin{array}{l} \partial_t \overline{W}_n^+ = -in(\delta_n s) \overline{W}_n^+ - n^2 \sin(2\pi k) \overline{Y}_n^+ - n^2 \sin\left(2\pi\left(k + \frac{\eta}{n}\right)\right) \overline{Y}_n^- \\ \quad - \gamma n^2 (2\overline{W}_n^+ - \overline{Y}_n^+ - \overline{Y}_n^-), \\ \partial_t \overline{Y}_n^+ = n^2 \sin(2\pi k) \overline{W}_n^+ - in^2(\sigma_n s) \overline{Y}_n^+ - n^2 \sin\left(2\pi\left(k + \frac{\eta}{n}\right)\right) \overline{W}_n^- \\ \quad - \gamma n^2 (2\overline{Y}_n^+ - \overline{W}_n^+ - \overline{W}_n^-), \\ \partial_t \overline{Y}_n^- = n^2 \sin\left(2\pi\left(k + \frac{\eta}{n}\right)\right) \overline{W}_n^+ + in^2(\sigma_n s) \overline{Y}_n^- - n^2 \sin(2\pi k) \overline{W}_n^- \\ \quad - \gamma n^2 (2\overline{Y}_n^- - \overline{W}_n^+ - \overline{W}_n^-), \\ \partial_t \overline{W}_n^- = in(\delta_n s) \overline{W}_n^- + n^2 \sin\left(2\pi\left(k + \frac{\eta}{n}\right)\right) \overline{Y}_n^+ + n^2 \sin(2\pi k) \overline{Y}_n^- \\ \quad - \gamma n^2 (2\overline{W}_n^- - \overline{Y}_n^+ - \overline{Y}_n^-), \end{array} \right. \quad (5.6)$$

5.3. Laplace transform of the dynamical system. We deduce from (5.5) an equation satisfied by \mathbf{w}_n - the four-dimensional vector of Laplace transforms of the Wigner functions defined by $\mathbf{w}_n := [w_n^+, y_n^+, y_n^-, w_n^-]^T$. For the clarity sake we shall use the notation

$$\begin{aligned} \mathbf{1} &:= [1, 1, 1, 1]^T, \quad \mathbf{e} := [1, -1, -1, 1]^T, \\ \mathbf{w}_n^0 &:= [W_n^+(0), Y_n^+(0), Y_n^-(0), W_n^-(0)]^T, \\ \overline{\mathbf{w}}_n^0 &:= [\overline{W}_n^+, \overline{Y}_n^+, \overline{Y}_n^-, \overline{W}_n^-]^T, \quad \widetilde{\mathbf{w}}_n^0 := [\widetilde{W}_n^+, \widetilde{Y}_n^+, \widetilde{Y}_n^-, \widetilde{W}_n^-]^T, \end{aligned}$$

and from (4.15) we have $\mathbf{w}_n^0 = \overline{\mathbf{w}}_n^0 + \widetilde{\mathbf{w}}_n^0$. As before, we shall often drop the variables (λ, η, k) from the notations. Taking the Laplace transform of both sides of (5.5) we obtain a linear system that can be written for any (λ, η, k) in the form

$$(\mathbf{M}_n \mathbf{w}_n)(\lambda, \eta, k) = \mathbf{w}_n^0(\eta, k) + \gamma n^2 \mathcal{I}_n(\lambda, \eta) \mathbf{e}, \quad (5.7)$$

where \mathcal{I}_n is defined as the scalar product

$$\mathcal{I}_n(\lambda, \eta) := \mathbf{e} \cdot \left[\mathbf{w}_n(\lambda, \eta, \cdot) \right]_n = \left[(w_n^+ - y_n^+ - y_n^- + w_n^-)(\lambda, \eta, \cdot) \right]_n.$$

In addition, the 2×2 block matrix $\mathbf{M}_n := \mathbf{M}_n(\lambda, \eta, k)$ is defined as follows:

$$\mathbf{M}_n := \begin{bmatrix} A_n & -n^2 \gamma_n^- \text{Id}_2 \\ -n^2 \gamma_n^+ \text{Id}_2 & B_n \end{bmatrix}, \quad (5.8)$$

where, given a positive integer N , Id_N denotes the $N \times N$ identity matrix, and A_n, B_n are 2×2 matrices:

$$A_n := \begin{bmatrix} a_n & -n^2\gamma^- \\ -n^2\gamma^+ & b_n \end{bmatrix}, \quad B_n := \begin{bmatrix} b_n^* & -n^2\gamma^- \\ -n^2\gamma^+ & a_n^* \end{bmatrix}.$$

Here

$$\begin{cases} a_n := \lambda + in(\delta_n s) + 2\gamma n^2, \\ b_n := \lambda + in^2(\sigma_n s) + 2\gamma n^2, \end{cases} \quad \begin{cases} \gamma_n^\pm := \gamma \pm \sin(2\pi(k + \frac{n}{n})), \\ \gamma^\pm := \gamma \pm \sin(2\pi k). \end{cases} \quad (5.9)$$

An elementary observation yields

$$\gamma_n^\pm(-\eta, -k) = \gamma_n^\mp(\eta, k), \quad \gamma^\pm(-\eta, -k) = \gamma^\mp(\eta, k), \quad (5.10)$$

By the linearity of the Laplace transform we can write

$$\mathfrak{w}_n = \overline{\mathfrak{w}}_n + \tilde{\mathfrak{w}}_n, \quad (5.11)$$

where $\overline{\mathfrak{w}}_n$ is the Laplace transform of $(\overline{W}_n^+(t), \overline{Y}_n^+(t), \overline{Y}_n^-(t), \overline{W}_n^-(t))$, and $\tilde{\mathfrak{w}}_n$ is the Laplace transform of $(\tilde{W}_n^+(t), \tilde{Y}_n^+(t), \tilde{Y}_n^-(t), \tilde{W}_n^-(t))$.

Performing the Laplace transform of both sides of (5.6) we conclude that $\overline{\mathfrak{w}}_n$ solves the equation

$$\mathbf{M}_n \overline{\mathfrak{w}}_n = \overline{\mathfrak{v}}_n^0 = \overline{W}_n^+ \mathbf{1}. \quad (5.12)$$

Using (5.7) we conclude that $\tilde{\mathfrak{w}}_n$ solves

$$\mathbf{M}_n \tilde{\mathfrak{w}}_n = \tilde{\mathfrak{v}}_n^0 + \gamma n^2 \mathcal{I}_n \mathbf{e}. \quad (5.13)$$

Following (5.11) we also write $\mathcal{I}_n = \overline{\mathcal{I}}_n + \tilde{\mathcal{I}}_n$, where

$$\overline{\mathcal{I}}_n(\lambda, \eta) := \mathbf{e} \cdot \left[\overline{\mathfrak{w}}_n(\lambda, \eta, \cdot) \right]_n \quad \text{and} \quad \tilde{\mathcal{I}}_n(\lambda, \eta) := \mathbf{e} \cdot \left[\tilde{\mathfrak{w}}_n(\lambda, \eta, \cdot) \right]_n.$$

In Appendix 7.1, we show that the matrix \mathbf{M}_n is invertible, therefore we can rewrite:

$$\overline{\mathfrak{w}}_n = \overline{W}_n^+ \mathbf{M}_n^{-1} \mathbf{1}, \quad (5.14)$$

$$\tilde{\mathfrak{w}}_n = \mathbf{M}_n^{-1} \tilde{\mathfrak{v}}_n^0 + \gamma n^2 \mathcal{I}_n \mathbf{M}_n^{-1} \mathbf{e}. \quad (5.15)$$

In Section 6 we study the contribution of the terms appearing in the right-hand sides of both (5.14) and (5.15) that reflect upon the evolution of the mechanical and fluctuating components of the energy functional.

6. PROOF OF THE HYDRODYNAMIC BEHAVIOR OF THE ENERGY

In this section we conclude the proof of Theorem 3.8, up to technical lemmas that are proved in Section 7.

6.1. Mechanical energy $\overline{\mathbf{w}}_n$. We start with the recollection of the results concerning the mechanical energy. The Laplace transform $\overline{\mathbf{w}}_n$ is autonomous from the thermal part and satisfies (5.14). Let us introduce, for any $\lambda > 0$ and $\eta \in \mathbb{Z}$, the *mechanical Laplace-Wigner function*

$$\mathcal{W}_{\text{mech}}^+(\lambda, \eta) := \sum_{\xi \in \mathbb{Z}} \frac{W(r_0; \eta, \xi)}{\frac{2\pi^2}{\gamma} [\xi^2 + (\xi + \eta)^2] + \lambda}. \quad (6.1)$$

From Lemma 5.1, it follows that $\mathcal{W}_{\text{mech}}^+(\lambda, \eta)$ is the Fourier-Laplace transform of the mechanical energy density $e_{\text{mech}}(t, u) = \frac{1}{2} (r(t, u))^2$, where $r(t, u)$ is the solution of (3.16).

Given $M \in \mathbb{N}$ we denote by \mathcal{P}_M the subspace of $\mathcal{C}^\infty(\mathbb{T} \times \mathbb{T})$ consisting of all trigonometric polynomials that are finite linear combinations of $e^{2\pi i \eta u} e^{2\pi i \xi v}$, with $\eta \in \{-M, \dots, M\}$, $\xi \in \mathbb{Z}$ and $u, v \in \mathbb{T}$.

Proposition 6.1 (Mechanical part). *For any $M \in \mathbb{N}$ there exists $\lambda_M > 0$ such that for any $G \in \mathcal{P}_M$ and $\lambda > \lambda_M$ we have*

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{\eta \in \mathbb{Z}} \left[\overline{\mathbf{w}}_n(\lambda, \eta, \cdot) (\mathcal{F}G)^*(\eta, \cdot) \right]_n &= \left(\sum_{\eta, \xi \in \mathbb{Z}} \frac{W(r_0; \eta, \xi)}{\frac{2\pi^2}{\gamma} [\xi^2 + (\xi + \eta)^2] + \lambda} (\mathcal{F}G)^*(\eta, 0) \right) \mathbf{1} \\ &= \left(\frac{1}{2} \int_{\mathbb{T}} \int_0^{+\infty} e^{-\lambda t} (r(t, u))^2 G^*(u, 0) dt du \right) \mathbf{1}. \end{aligned} \quad (6.2)$$

Moreover, for any $\eta \in \{-M, \dots, M\}$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left\{ \gamma n^2 \overline{\mathcal{I}}_n(\lambda, \eta) \right\} &= \frac{4\pi^2}{\gamma} \sum_{\xi \in \mathbb{Z}} \frac{\xi(\xi + \eta) W(r_0; \eta, \xi)}{\frac{2\pi^2}{\gamma} [\xi^2 + (\xi + \eta)^2] + \lambda} \\ &= \frac{1}{2\gamma} \mathcal{L} \left(\mathcal{F}((\partial_u r)^2)(\eta) \right) (\lambda). \end{aligned} \quad (6.3)$$

The proof of Proposition 6.1 is presented in Section 7.3.

6.2. The closing of thermal energy equation. We now analyze equation (5.15). After averaging (5.15) over $k \in \widehat{\mathbb{T}}_n$ and scalarly multiplying by \mathbf{e} , we obtain the equation:

$$\tilde{\mathcal{I}}_n = \tilde{z}_n^{(0)} + \gamma n^2 (\tilde{\mathcal{I}}_n + \overline{\mathcal{I}}_n) \mathcal{M}_n, \quad (6.4)$$

where

$$\begin{aligned} \tilde{z}_n^{(0)} &:= \left[\mathbf{e} \cdot (\mathbf{M}_n^{-1} \tilde{\mathbf{w}}_n^0) \right]_n \\ \mathcal{M}_n &:= \left[\mathbf{e} \cdot (\mathbf{M}_n^{-1} \mathbf{e}) \right]_n. \end{aligned}$$

Therefore from (6.4) we solve explicitly

$$\tilde{\mathcal{I}}_n(\lambda, \eta) = \frac{n^2 \tilde{z}_n^{(0)}(\lambda, \eta) + (\gamma n^2 \bar{\mathcal{I}}_n(\lambda, \eta)) (n^2 \mathcal{M}_n(\lambda, \eta))}{n^2 (1 - \gamma n^2 \mathcal{M}_n(\lambda, \eta))}. \quad (6.5)$$

Asymptotics of $n^2 \bar{\mathcal{I}}_n(\lambda, \eta)$ is given in (6.3). Below we describe the terms $n^2 \tilde{z}_n^{(0)}(\lambda, \eta)$ and $n^2 \mathcal{M}_n(\lambda, \eta)$ that also appear in the right hand side (6.5).

Lemma 6.2. *Fix $M \in \mathbb{N}$. There exists $\lambda_M > 0$ such that, for any $\lambda > \lambda_M$ and $\eta \in \{-M, \dots, M\}$*

$$\lim_{n \rightarrow \infty} \left\{ \gamma n^2 \mathcal{M}_n(\lambda, \eta) \right\} = 1, \quad (6.6)$$

$$\lim_{n \rightarrow \infty} \left\{ n^2 (1 - \gamma n^2 \mathcal{M}_n(\lambda, \eta)) \right\} = \frac{1}{2\gamma} \left(\lambda + \frac{\eta^2 \pi^2}{\gamma} \right), \quad (6.7)$$

and

$$\lim_{n \rightarrow \infty} \left\{ \gamma n^2 \tilde{z}_n^{(0)}(\lambda, \eta) \right\} = (\mathcal{F}e_{\text{thm}}(0, \cdot))(\eta). \quad (6.8)$$

As a direct consequence of the above lemma and (6.3), we have:

Corollary 6.3. *Fix $M \in \mathbb{N}$. There exists $\lambda_M > 0$ such that, for any $\lambda > \lambda_M$, $\eta \in \{-M, \dots, M\}$*

$$\lim_{n \rightarrow \infty} \tilde{\mathcal{I}}_n(\lambda, \eta) = 2\mathcal{W}_{\text{thm}}^+(\lambda, \eta),$$

where

$$\mathcal{W}_{\text{thm}}^+(\lambda, \eta) := \left(\lambda + \frac{\eta^2 \pi^2}{\gamma} \right)^{-1} \left\{ (\mathcal{F}e_{\text{thm}}(0, \cdot))(\eta) + \frac{1}{2\gamma} \mathcal{L} \left(\mathcal{F}((\partial_u r)^2)(\eta) \right)(\lambda) \right\}.$$

The following lemma finalizes the identification of the limit for the Fourier transform of the thermal energy:

Proposition 6.4. *Fix $M \in \mathbb{N}$. There exists $\lambda_M > 0$ such that, for any $\lambda > \lambda_M$, $\eta \in \{-M, \dots, M\}$*

$$\lim_{n \rightarrow \infty} \left\{ \tilde{\mathcal{I}}_n(\lambda, \eta) - 2 [\tilde{w}_n^+(\lambda, \eta, \cdot)]_n \right\} = 0. \quad (6.9)$$

The proofs of Lemma 6.2 and Proposition 6.4 go very much along the lines of the arguments presented in Section 7 and we will not present the details here. They are basically consequences of the following limit

$$\lim_{n \rightarrow \infty} \left\{ n^2 \mathbf{e}_1 \cdot \mathbf{M}_n^{-1}(\lambda, \eta, k) \mathbf{e} \right\} = \frac{1}{2\gamma}.$$

6.3. Asymptotics of $\tilde{\mathbf{w}}_n$ and \mathbf{w}_n . With a little more work one can prove the following *local equilibrium* result, which is an easy consequence of Proposition 6.1, Corollary 6.3 and Proposition 6.4 (recall also (5.15)).

Theorem 6.5. *Fix $M \in \mathbb{N}$. There exists $\lambda_M > 0$ such that, for any $\lambda > \lambda_M$ and $G \in \mathcal{P}_M$ we have*

$$\begin{aligned} \lim_{n \rightarrow +\infty} \sum_{\eta \in \mathbb{Z}} \left[w_n^+(\lambda, \eta, \cdot) (\mathcal{F}G)^*(\eta, \cdot) \right]_n \\ = \sum_{\eta \in \mathbb{Z}} \left\{ \mathcal{W}_{\text{thm}}^+(\lambda, \eta) \int_{\mathbb{T}} (\mathcal{F}G)^*(\eta, v) \, dv + \mathcal{W}_{\text{mech}}^+(\lambda, \eta) (\mathcal{F}G)^*(\eta, 0) \right\} \end{aligned} \quad (6.10)$$

and

$$\lim_{n \rightarrow +\infty} \sum_{\eta \in \mathbb{Z}} \left[y_n^+(\lambda, \eta, \cdot) (\mathcal{F}G)^*(\eta, \cdot) \right]_n = \sum_{\eta \in \mathbb{Z}} \mathcal{W}_{\text{mech}}^+(\lambda, \eta) (\mathcal{F}G)^*(\eta, 0), \quad (6.11)$$

We will not give the details for the proof of this last theorem, since the argument is very similar to Proposition 6.1.

6.4. End of the proof of Theorem 3.8. From the uniform bound (4.13), we know that the sequence of all Wigner distributions $\{\mathbf{W}_n^+(\cdot), \mathbf{Y}_n^+(\cdot), \mathbf{Y}_n^-(\cdot), \mathbf{W}_n^-(\cdot)\}_n$ is sequentially pre-compact with respect to the \star -weak topology in the dual space of $\mathbf{L}^1(\mathbb{R}_+, \mathcal{A}_0)$. More precisely, one can choose a subsequence n_m such that any of the components above, say for instance $\mathbf{W}_{n_m}^+(\cdot)$, \star -weakly converges in the dual space of $\mathbf{L}^1(\mathbb{R}_+, \mathcal{A}_0)$ to some $\mathbf{W}^+(\cdot)$.

To characterize its limit, we consider $\mathbf{w}_{n_m}^+(\lambda)$ obtained by taking the Laplace transforms of the respective $\mathbf{W}_{n_m}^+(\cdot)$. For any $\lambda > 0$, it converges \star -weakly, as $n_m \rightarrow +\infty$, in \mathcal{A}'_0 to some $\mathbf{w}^+(\lambda)$ that is the Laplace transform of $\mathbf{W}^+(\cdot)$. The latter is defined as

$$\langle \mathbf{w}^+(\lambda), G \rangle := \int_0^\infty \langle \mathbf{W}^+(t), e^{-\lambda t} G \rangle \, dt \quad \lambda > 0, \, G \in \mathcal{A}_0.$$

Given a trigonometric polynomial $G \in \mathcal{C}^\infty(\mathbb{T} \times \mathbb{T})$ we conclude, thanks to Theorem 6.5, that for any $\lambda > \lambda_M$,

$$\langle \mathbf{w}^+(\lambda), G \rangle = \int_{\mathbb{R}_+ \times \mathbb{T}^2} e^{-\lambda t} e(t, u) G^*(u, v) \, dt \, du \, dv, \quad (6.12)$$

where $e(t, u)$ is defined as in Theorem 3.8 and $M \in \mathbb{N}$ is such that $\mathcal{F}G(\eta, v) \equiv 0$ for all $|\eta| > M$.

Due to the uniqueness of the Laplace transform (that can be argued by analytic continuation), this proves that in fact equality (6.12) holds for all $\lambda > 0$. By a density argument it can be then extended to all $G \in \mathcal{A}_0$ and shows that $\mathbf{W}^+(t, u, v) = e(t, u)$, for any $(t, u, v) \in \mathbb{R}_+ \times \mathbb{T}^2$. This ends the proof of (3.19), and thus Theorem 3.8.

7. PROOFS OF THE TECHNICAL RESULTS STATED IN SECTION 6

In what follows we shall adopt the following notation: we say that the sequence $C_n(\lambda, \eta, k) \leq 1$ if for any given integer $M \in \mathbb{N}$, there exist $\lambda_M > 0$ and $n_M \in \mathbb{N}$ such that

$$\sup \left\{ C_n(\lambda, \eta, k) ; \lambda > \lambda_M, \eta \in \{-M, \dots, M\}, n > n_M, k \in \widehat{\mathbb{T}}_n \right\} < +\infty.$$

7.1. Invertibility of $\mathbf{M}_n(\lambda, \eta, k)$.

Proposition 7.1. *The matrix $\mathbf{M}_n(\lambda, \eta, k)$ defined in (5.8) is invertible for all $n \geq 1$, $\lambda > 0$ and $(\eta, k) \in \mathbb{Z} \times \widehat{\mathbb{T}}_n$.*

Proof. The block entries of the matrix \mathbf{M}_n defined in (5.9) satisfy the commutation relation

$$[A_n, B_n] = A_n B_n - B_n A_n = \begin{bmatrix} 0 & -2\gamma^- n^2 \operatorname{Re}[a_n - b_n] \\ -2\gamma^+ n^2 \operatorname{Re}[b_n - a_n] & 0 \end{bmatrix} = 0.$$

Thanks to the well known formula for the determinants of block matrices with commuting entries we have (see e.g. [4, formula (Ib), p. 46])

$$\det(\mathbf{M}_n) = \det(A_n B_n - \gamma_n^+ \gamma_n^- n^4 \operatorname{Id}_2) = |a_n b_n^* + n^3 (\delta \gamma_n)|^2 - 4n^4 \gamma^+ \gamma^- (\operatorname{Re}[a_n])^2$$

and, substituting from (5.9), we get

$$\begin{aligned} \det(\mathbf{M}_n) &= n^6 \left((\delta_n s) (\sigma_n s) + (\delta \gamma_n) \right)^2 \\ &\quad + (\lambda + 2\gamma n^2)^2 \left\{ \lambda^2 + n^2 (4\gamma \lambda + (\delta_n s)^2) \right. \\ &\quad \left. + n^4 \left(2 \sin^2(2\pi k) + 2 \sin^2 \left(2\pi \left(k + \frac{\eta}{n} \right) \right) + (\sigma_n s)^2 \right) \right\}. \end{aligned} \quad (7.1)$$

Here $\delta_n s$, $\sigma_n s$ are given by (5.4) and

$$\delta \gamma_n(\eta, k) := n(\gamma^+ \gamma^- - \gamma_n^+ \gamma_n^-) = n \left(\sin^2 \left(2\pi \left(k + \frac{\eta}{n} \right) \right) - \sin^2(2\pi k) \right).$$

The proposition is a direct conclusion of (7.1). \square

It is also clear that

$$\det(\mathbf{M}_n) = n^8 \Delta_n, \quad (7.2)$$

where

$$\Delta_n = \frac{1}{n^2} \left\{ 4\gamma^2 \Gamma_n + 4\gamma^2 (4\lambda\gamma + (\delta_n s)^2) + ((\delta_n s)(\sigma_n s) + (\delta \gamma_n))^2 \right\} + 4\gamma \lambda \frac{\Gamma_n}{n^4} + \frac{C_n}{n^3}. \quad (7.3)$$

for some $|C_n| \leq 1$ and

$$\Gamma_n(\eta, k) := n^2 \left(2 \sin^2(2\pi k) + 2 \sin^2 \left(2\pi \left(k + \frac{\eta}{n} \right) \right) + (\sigma_n s)^2 \right). \quad (7.4)$$

On the one hand, note that for k sufficiently *far* from 0, the dominant term is $\frac{4\gamma^2 \Gamma_n}{n^2}$ and then

$$\Delta_n \sim 4\gamma^2 (4 \sin^2(2\pi k) + 16 \sin^4(\pi k)).$$

On the other hand, for $k = \frac{\xi}{n}$ and fixed $\xi \in \mathbb{Z}$ we have

$$n^2 \Delta_n \left(\lambda, \eta, \frac{\xi}{n} \right) = \frac{1}{n^6} \det(\mathbf{M}_n) \left(\lambda, \eta, \frac{\xi}{n} \right) \sim 16\gamma^2 \left[\lambda\gamma + 2\pi^2 (\xi^2 + (\eta + \xi)^2) \right]. \quad (7.5)$$

Since the block entries of \mathbf{M}_n commute we can also write

$$\mathbf{M}_n^{-1} = \begin{bmatrix} \left[A_n B_n - (\gamma_n^+ \gamma_n^- n^4) \text{Id}_2 \right]^{-1} & 0 \\ 0 & \left[A_n B_n - (\gamma_n^+ \gamma_n^- n^4) \text{Id}_2 \right]^{-1} \end{bmatrix} \begin{bmatrix} B_n & \gamma_n^- n^2 \text{Id}_2 \\ \gamma_n^+ n^2 \text{Id}_2 & A_n \end{bmatrix}.$$

Note that

$$\left[A_n B_n - (\gamma_n^+ \gamma_n^- n^4) \text{Id}_2 \right]^{-1} = \frac{1}{\det(\mathbf{M}_n)} \begin{bmatrix} a_n^* b_n + n^3 (\delta\gamma_n) & 2\gamma^- n^2 \text{Re}[a_n] \\ 2\gamma^+ n^2 \text{Re}[b_n] & a_n b_n^* + n^3 (\delta\gamma_n) \end{bmatrix}.$$

With these formulas we conclude that

$$\mathbf{M}_n^{-1} = \frac{1}{\det(\mathbf{M}_n)} \begin{bmatrix} d_n^+ & \gamma^- d_n & \gamma_n^- c_n^* & \gamma^- \gamma_n^- c_n^0 \\ \gamma^+ d_n^0 & d_n^- & \gamma^+ \gamma_n^- c_n^0 & \gamma_n^- c_n \\ \gamma_n^+ c_n^* & \gamma^- \gamma_n^+ c_n^0 & (d_n^-)^* & \gamma^- (d_n^0)^* \\ \gamma^+ \gamma_n^+ c_n^0 & \gamma_n^+ c_n & \gamma^+ d_n^* & (d_n^+)^* \end{bmatrix} \quad (7.6)$$

where all the constants are explicit and given by

$$\begin{cases} d_n^+ := a_n^* |b_n|^2 + n^3 b_n^* (\delta\gamma_n) - 2\gamma^- \gamma^+ n^4 \text{Re}[a_n], \\ d_n^- := b_n^* |a_n|^2 + n^3 a_n^* (\delta\gamma_n) - 2\gamma^- \gamma^+ n^4 \text{Re}[b_n], \\ d_n := 2n^2 a_n^* \text{Re}[a_n] - n^2 a_n^* b_n - n^5 (\delta\gamma_n), \\ d_n^0 := 2n^2 b_n^* \text{Re}[b_n] - n^2 a_n b_n^* - n^5 (\delta\gamma_n), \\ c_n := n^2 a_n b_n^* + n^5 (\delta\gamma_n), \\ c_n^0 := 2n^4 \text{Re}[a_n]. \end{cases} \quad (7.7)$$

7.2. Asymptotics of the coefficients. Substituting from (5.9) into the respective formulas of (7.7) and then identifying the order of magnitude of the appearing terms we conclude the following:

Lemma 7.2. *The following asymptotic equalities hold:*

$$\frac{d_n^+}{n^6} = 4\gamma^3 + 4\gamma \sin^2(2\pi k) + 2\gamma(\sigma_n s)^2 - \frac{i(\delta_n s)}{n} (4\gamma^2 + (\sigma_n s)^2) + \frac{C_n}{n^2},$$

$$\frac{c_n}{n^6} = 4\gamma^2 - 2i\gamma(\sigma_n s) + \sin^2(2\pi(k + \frac{\eta}{n})) - \sin^2(2\pi k) + \frac{(\delta_n s)(\sigma_n s)}{n} + \frac{C_n}{n^2},$$

$$\frac{d_n}{n^6} = 4\gamma^2 - 2i\gamma(\sigma_n s) + \sin^2(2\pi k) - \sin^2(2\pi(k + \frac{\eta}{n})) - 2i\gamma \frac{(\delta_n s)}{n} - \frac{(\delta_n s)(\sigma_n s)}{n} + \frac{C_n}{n^2},$$

$$\frac{c_n^0}{n^6} = 4\gamma + \frac{C_n}{n^2},$$

$$\frac{d_n^0}{n^6} = 4\gamma^2 - 2i\gamma(\sigma_n s) + \sin^2(2\pi k) - \sin^2(2\pi(k + \frac{\eta}{n})) - 2i\gamma \frac{(\delta_n s)}{n} - \frac{(\delta_n s)(\sigma_n s)}{n} + \frac{C_n}{n^2},$$

$$\frac{d_n^-}{n^6} = 4\gamma^3 + 4\gamma \sin^2(2\pi k) - 4i\gamma^2(\sigma_n s) + 2\gamma \left(\sin^2(2\pi(k + \frac{\eta}{n})) - \sin^2(2\pi k) \right) + \frac{C_n}{n^2},$$

with $C_n \leq 1$.

From the above asymptotics it is clear that $\mathbf{M}_n^{-1}(\lambda, \eta, k) \rightarrow 0$ for a fixed $k \neq 0$. In addition, $\mathbf{M}_n^{-1}(\lambda, \eta, \frac{\xi}{n})$ and $n^2 \mathbf{e}^T \mathbf{M}_n^{-1}(\lambda, \eta, k)$ tend to finite limits that we need to compute explicitly in order to complete the proof.

Using (7.6), (7.2) and the formulas for the asymptotics of the entries of \mathbf{M}_n^{-1} , provided by Lemma 7.2, we conclude that

$$\lim_{n \rightarrow \infty} \mathbf{M}_n^{-1}\left(\lambda, \eta, \frac{\xi}{n}\right) = \frac{\gamma}{4\lambda\gamma + 8\pi^2 [\xi^2 + (\xi + \eta)^2]} \mathbf{1} \otimes \mathbf{1}. \quad (7.8)$$

The above in particular implies that $\mathbf{e}^T \cdot \mathbf{M}_n^{-1}(\lambda, \eta, \frac{\xi}{n}) \mathbf{1} \rightarrow 0$, for any $\xi \in \mathbb{Z}$. By the same token we can also compute

$$\lim_{n \rightarrow +\infty} \left\{ n^2 \mathbf{e}^T \cdot \mathbf{M}_n^{-1}\left(\lambda, \eta, \frac{\xi}{n}\right) \mathbf{1} \right\} = \frac{4\pi^2 \xi(\xi + \eta)}{\lambda\gamma^2 + 2\gamma\pi^2 [\xi^2 + (\xi + \eta)^2]}. \quad (7.9)$$

and

$$\lim_{n \rightarrow \infty} \left\{ n^2 \mathbf{e}^T \cdot \mathbf{M}_n^{-1}(\lambda, \eta, k) \right\} = \frac{1}{2\gamma} [1, 0, 0, 1], \quad k \neq 0. \quad (7.10)$$

We prove here only (7.9) and we let the reader verify (7.8) and (7.10) using similar computations. An explicit calculation gives

$$n^2 \mathbf{e}^T \cdot (\mathbf{M}_n^{-1} \mathbf{1}) = \frac{n^2 \Xi_n}{\det(\mathbf{M}_n)}, \quad (7.11)$$

where

$$\begin{aligned}\Xi_n(\lambda, \eta, k) &:= 2\operatorname{Re}[d_n^+ - d_n^-] + \gamma^-(d_n - (d_n^0)^\star) + \gamma^+(d_n^\star - d_n^0) \\ &\quad + 4i \sin\left(2\pi\left(k + \frac{\eta}{n}\right)\right) \operatorname{Im}[c_n] + 4 \sin(2\pi k) \sin\left(2\pi\left(k + \frac{\eta}{n}\right)\right) c_n^0.\end{aligned}$$

Substituting from (7.7) yields:

$$\begin{aligned}\Xi_n(\lambda, \eta, k) &= (\lambda + 2\gamma n^2) \left\{ 2[n^4(\sigma_n s)^2 - n^2(\delta_n s)^2] \right. \\ &\quad + 4in^4 \left(\sin(2\pi k) - \sin\left(2\pi\left(k + \frac{\eta}{n}\right)\right) \right) (\sigma_n s) \\ &\quad + 4in^3 \left(\sin\left(2\pi\left(k + \frac{\eta}{n}\right)\right) + \sin(2\pi k) \right) (\delta_n s) \\ &\quad \left. + 8n^4 \sin(2\pi k) \sin\left(2\pi\left(k + \frac{\eta}{n}\right)\right) \right\}.\end{aligned}\tag{7.12}$$

From the above and a direct calculation we obtain (7.9).

7.3. Proof of Proposition 6.1. Basically the argument follows the same idea as the proof of Proposition 4.1. In fact (7.8) implies that $\overline{\mathbf{w}}_n(\lambda, \eta, k)$ concentrates on small k -s like $\overline{W}_n^+(\eta, k)$. The main difficulty is to deal with the averaging $[\cdot]_n$. For that purpose, for $\rho \in (0, \frac{1}{2})$, define

$$\widehat{\mathbb{T}}_{n,\rho} := \left\{ k \in \widehat{\mathbb{T}}_n : |\sin(\pi k)| \geq n^{-\rho} \right\}\tag{7.13}$$

and its complement $\widehat{\mathbb{T}}_{n,\rho}^c := \widehat{\mathbb{T}}_n \setminus \widehat{\mathbb{T}}_{n,\rho}$. Recall the left-hand side of (6.2): it can be written as $\mathbf{I}_n + \mathbf{II}_n$, where \mathbf{I}_n and \mathbf{II}_n correspond to the summations over $\widehat{\mathbb{T}}_{n,\rho}$ and $\widehat{\mathbb{T}}_{n,\rho}^c$ respectively. First we show that for $G \in \mathcal{P}_M$

$$\mathbf{I}_n = \sum_{\eta \in \mathbb{Z}} \frac{1}{n} \sum_{k \in \widehat{\mathbb{T}}_{n,\rho}} (\mathbf{M}_n^{-1}(\lambda, \eta, k) \mathbf{1}) \overline{W}_n^+(\eta, k) (\mathcal{F}G)^\star(\eta, k) \xrightarrow{n \rightarrow \infty} 0.\tag{7.14}$$

In fact we have

$$\begin{aligned}\left\| \sum_{\eta \in \mathbb{Z}} \frac{1}{n} \sum_{|\xi| \geq n^\rho} \left(\mathbf{M}_n^{-1}\left(\lambda, \eta, \frac{\xi}{n}\right) \mathbf{1} \right) \overline{W}_n^+\left(\eta, \frac{\xi}{n}\right) (\mathcal{F}G)^\star\left(\eta, \frac{\xi}{n}\right) \right\|_\infty \\ \leq C \|G\|_0 \sum_{|\eta| \leq M} \sum_{|\xi| \geq n^\rho} \left\| \mathbf{M}_n^{-1}\left(\lambda, \eta, \frac{\xi}{n}\right) \mathbf{1} \right\|_\infty \xrightarrow{n \rightarrow \infty} 0,\end{aligned}$$

where the constant C depends only on the initial mechanical energy. Then by the same argument as the one used in the proof of Proposition 4.1, and from (4.26)

and (7.8) we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{|\eta| \leq M} \frac{1}{n} \sum_{|\xi| < n^\rho} \left(\mathbf{M}_n^{-1} \left(\lambda, \eta, \frac{\xi}{n} \right) \mathbf{1} \right) \overline{W}_n^+ \left(\eta, \frac{\xi}{n} \right) (\mathcal{F}G)^* \left(\eta, \frac{\xi}{n} \right) \\ = \sum_{\eta, \xi \in \mathbb{Z}} \frac{\gamma W(r_0; \eta, \xi)}{\lambda \gamma + 2\pi^2 [\xi^2 + (\xi + \eta)^2]} (\mathcal{F}G)^*(\eta, 0) \mathbf{1}. \end{aligned}$$

This concludes the proof of (6.2). Concerning the proof of (6.3), recall that

$$\overline{\mathcal{I}}_n = \left[e \cdot (\mathbf{M}_n^{-1} \mathbf{1}) \overline{W}_n^+ \right]_n,$$

and that we have already computed the limit (7.9). Consequently, the result will follow if we are able to show that the contribution to the k -averaging from the higher frequencies is negligible. The quantity $n^2 \overline{\mathcal{I}}_n$ can be written as $\mathbf{I}_n + \mathbf{II}_n$, where \mathbf{I}_n and \mathbf{II}_n correspond to the summations over $\widehat{\mathbb{T}}_{n,\rho}$ and $\widehat{\mathbb{T}}_{n,\rho}^c$ respectively. Using the explicit computations (7.11), (7.12) and (7.3) we can write

$$\begin{aligned} n^2 e \cdot (\mathbf{M}_n^{-1} \mathbf{1}) - \gamma^{-1} &= \gamma^{-1} \left\{ 2 \left(\sin(2\pi k) - \sin \left(2\pi \left(k + \frac{\eta}{n} \right) \right) \right)^2 + \frac{C_n}{n} \right\} \\ &\times \left\{ 2 \sin^2(2\pi k) + 2 \sin^2 \left(2\pi \left(k + \frac{\eta}{n} \right) \right) + (\sigma_n s)^2 + \frac{C'_n}{n^2} \right\}^{-1}. \end{aligned}$$

It is clear from the above equality that

$$\lim_{n \rightarrow +\infty} \sup_{|\eta| \leq M} \sup_{k \in \widehat{\mathbb{T}}_{n,\rho}} \left| n^2 e \cdot (\mathbf{M}_n^{-1} \mathbf{1})(\lambda, \eta, k) - \frac{1}{\gamma} \right| = 0. \quad (7.15)$$

Thanks to (7.15) we conclude that $\lim_{n \rightarrow +\infty} (\mathbf{I}_n - \mathbf{I}'_n) = 0$, where

$$\mathbf{I}'_n := \frac{1}{\gamma n} \sum_{k \in \widehat{\mathbb{T}}_{n,\rho}} \overline{W}_n^+(\eta, k). \quad (7.16)$$

After a straightforward calculation using the definition of $\overline{W}_n^+(\eta, k)$ (see (4.16) and (4.18)) we conclude that (7.16) equals

$$\frac{1}{2\gamma} \sum_{\xi, \xi' \in \mathbb{Z}} \sum_{k \in \widehat{\mathbb{T}}_{n,\rho}} 1_{\mathbb{Z}} \left(k - \frac{\xi'}{n} \right) 1_{\mathbb{Z}} \left(-k + \frac{\xi - \eta}{n} \right) (\mathcal{F}r_0)(\xi) (\mathcal{F}r_0)^*(\xi'),$$

where $1_{\mathbb{Z}}$ is the indicator function of the integer lattice. Due to the assumed separation of k from 0, see (7.13), and the decay of the Fourier coefficients of $r_0(\cdot)$ (that belongs to $\mathcal{C}^\infty(\mathbb{T})$) we conclude from the above that $\lim_{n \rightarrow +\infty} \mathbf{I}'_n = 0$, thus also $\lim_{n \rightarrow +\infty} \mathbf{I}_n = 0$.

Moreover, a similar calculation also yields

$$\mathbf{II}_n = \frac{1}{2} \sum_{\xi \in \mathcal{N}_{\rho,n}} n^2 e \cdot (\mathbf{M}_n^{-1} \mathbf{1}) \left(\lambda, \eta, \frac{\xi}{n} \right) (\mathcal{F}r_0)(\xi + \eta) (\mathcal{F}r_0)^*(\xi),$$

where

$$\mathcal{N}_{\rho,n} := \left\{ \xi \in \mathbb{Z} : |\xi| \leq \frac{n}{2}, \quad \left| \sin\left(\frac{\pi\xi}{n}\right) \right| \leq n^{-\rho} \right\}.$$

Using the dominated convergence theorem we conclude from (7.9) that

$$\lim_{n \rightarrow +\infty} \Pi_n = \sum_{\xi \in \mathbb{Z}} \frac{4\pi^2 \xi(\xi + \eta) W(r_0; \eta, \xi)}{\lambda\gamma^2 + 2\gamma\pi^2 [\xi^2 + (\xi + \eta)^2]}. \quad (7.17)$$

7.4. Proof of Lemma 6.2 and Proposition 6.4. Since (6.6) is a direct consequence of (6.7), we prove directly (6.7), that is a consequence of the following lemma.

Lemma 7.3. *The following asymptotic equality holds:*

$$\mathcal{S}_n(\lambda, \eta) := n^2 (1 - \gamma n^2 \mathcal{M}_n(\lambda, \eta)) = \frac{1}{2\gamma} \left(\lambda + \frac{\eta^2 \pi^2}{\gamma} \right) + \frac{C_n}{n}, \quad (7.18)$$

where $|C_n| \leq 1$.

Proof. After a direct calculation, we obtain

$$\mathcal{S}_n(\lambda, \eta) = \left[\frac{n^2}{\det(\mathbf{M}_n)} (\det(\mathbf{M}_n) - \gamma n^2 \Theta_n) \right]_n,$$

where

$$\Theta_n := 2\operatorname{Re} \left[d_n^+ + d_n^- + 2\gamma^2 c_n^0 - 2\gamma c_n \right] - \gamma^- (d_n + (d_n^0)^*) - \gamma^+ (d_n^* + d_n^0).$$

Therefore, from (7.7) and Lemma 7.2, we have

$$\mathcal{S}_n(\lambda, \eta) = \frac{\lambda}{2\gamma n} \sum_{k \in \widehat{\mathbb{T}}_n} \mathbf{I}_n(\eta, k) + \frac{1}{4\gamma^2 n} \sum_{k \in \widehat{\mathbb{T}}_n} \Pi_n(\eta, k).$$

where

$$\begin{aligned} \mathbf{I}_n &:= \frac{\Gamma_n + 2\lambda\gamma + (\delta_n s)^2 + \frac{C_n}{n}}{\Gamma_n + 4\lambda\gamma + (\delta_n s)^2 + \gamma^{-2}((\delta_n s)(\sigma_n s) + (\delta\gamma_n))^2 + \frac{C'_n}{n}}, \\ \Pi_n &:= \frac{n^2 ((\sigma_n s)(\delta_n s) + (\delta\gamma_n))^2}{\Gamma_n + 4\lambda\gamma + (\delta_n s)^2 + \gamma^{-2}((\delta_n s)(\sigma_n s) + (\delta\gamma_n))^2 + \frac{C'_n}{n}}. \end{aligned}$$

The expressions C_n, C'_n satisfy $|C_n| + |C'_n| \leq 1$. Directly from the definition of \mathbf{I}_n and Π_n we conclude that $|\mathbf{I}_n| + |\Pi_n| \leq 1$ and

$$\begin{aligned} \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{k \in \widehat{\mathbb{T}}_n} \mathbf{I}_n &= 1, \\ \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{k \in \widehat{\mathbb{T}}_n} \Pi_n &= (2\pi\eta)^2 \int_{\mathbb{T}} \frac{[4 \sin(2\pi v) \sin^2(\pi v) + \sin(4\pi v)]^2}{4 \sin^2(2\pi v) + 16 \sin^4(\pi v)} dv. \end{aligned}$$

Using trigonometric identities

$$2 \sin^2(\pi v) = 1 - \cos(2\pi v) \quad \text{and} \quad \sin(4\pi v) = 2 \sin(2\pi v) \cos(2\pi v)$$

we conclude that the last integral equals

$$\int_{\mathbb{T}} \frac{[2 \sin(2\pi v)(1 - \cos(2\pi v)) + \sin(4\pi v)]^2}{4 \sin^2(2\pi v) + 4[1 - \cos(2\pi v)]^2} dv = \int_{\mathbb{T}} \frac{\sin^2(2\pi v)}{2(1 - \cos(2\pi v))} dv = \frac{1}{2}.$$

Thus, we obtain

$$\mathcal{S}_n(\lambda, \eta) = \frac{1}{2\gamma} \left(\lambda + \frac{\pi^2 \eta^2}{\gamma} \right) + \frac{C_n}{n},$$

with $|C_n| \leq 1$. □

It remains to prove (6.8). This would be a direct consequence of (7.10), but we need some care in exchanging the limit with the $[\cdot]_n$ averaging.

Choose $\rho \in (0, 1)$, then we can decompose

$$n^2 \left[e \cdot (\mathbf{M}_n^{-1} \tilde{\mathbf{v}}_n^0) \right]_n = \frac{1}{2\gamma} \left(\left[\widetilde{W}_n^+ \right]_n + \left[\widetilde{W}_n^- \right]_n \right) + K_n^{(1)} + K_n^{(2)}$$

where

$$K_n^{(1)}(\lambda, \eta) = \frac{1}{n} \sum_{k \in \widehat{\mathbb{T}}_{n,\rho}} \left(e^T \mathbf{M}_n^{-1}(\lambda, \eta, k) - \frac{1}{2\gamma} u \right) \cdot \tilde{\mathbf{v}}_n^0(\eta, k)$$

with $u = [1, 0, 0, 1]^T$, and the definition of $K_n^{(2)}$ differs from $K_n^{(1)}$ only in that the range of the summation in k extends over $\widehat{\mathbb{T}}_{n,\rho}^c$.

From (4.22) we have

$$\lim_{n \rightarrow \infty} \left[\widetilde{W}_n^+(\eta, \cdot) \right]_n = \lim_{n \rightarrow \infty} \left[\widetilde{W}_n^-(\eta, \cdot) \right]_n = (\mathcal{F}e_{\text{thm}}(0, \cdot))(\eta),$$

and therefore we only have to prove that $K_n^{(1)}$ and $K_n^{(2)}$ vanish as $n \rightarrow \infty$. Concerning $K_n^{(1)}$ we write

$$\begin{aligned} |K_n^{(1)}(\lambda, \eta)| &\leq \sup_{k \in \widehat{\mathbb{T}}_{n,\rho}} \left\| e^T \mathbf{M}_n^{-1}(\lambda, \eta, k) - \frac{1}{2\gamma} u \right\|_{\infty} \left\| [\tilde{\mathbf{v}}_n^0(\eta, \cdot)]_n \right\|_{\infty} \\ &\leq C \sup_{k \in \widehat{\mathbb{T}}_{n,\rho}} \left\| e^T \mathbf{M}_n^{-1}(\lambda, \eta, k) - \frac{1}{2\gamma} u \right\|_{\infty}, \end{aligned} \quad (7.19)$$

where C depends on the bound on the initial energy. By direct estimation, using the information on the asymptotic behavior for the coefficients of \mathbf{M}_n^{-1} , provided by (7.10), we conclude that the right hand-side of (7.19) converges to 0 as $n \rightarrow \infty$, for any given λ and η .

Concerning $K_n^{(2)}$, since $n^2 e^T \cdot \mathbf{M}_n^{-1}(\lambda, \eta, k)$ are uniformly bounded in k , for any integer M there exists a constant $C_M > 0$ such that, for all n , $\lambda > \lambda_M$, $|\eta| \leq M$,

$$|K_n^{(2)}(\lambda, \eta)| \leq \frac{C_M}{n} \sum_{k \in \widehat{\mathbb{T}}_{n,\rho}^c} |\tilde{\mathbf{v}}_n^0(\eta, k)|.$$

Using the Cauchy-Schwarz inequality we have

$$\begin{aligned} |K_n^{(2)}(\lambda, \eta)| &\leq \frac{C'_M}{n} |\widehat{\mathbb{T}}_{n,\rho}^c|^{\frac{1}{2}} \left(\sum_{k \in \widehat{\mathbb{T}}_n} \mathbb{E} |\widetilde{\mathbf{v}}_n^0(\eta, k)|^2 \right)^{1/2} \\ &\leq C'_M \left(\frac{|\widehat{\mathbb{T}}_{n,\rho}^c|}{n} \right)^{1/2} w_*^{\frac{1}{2}} \xrightarrow{n \rightarrow \infty} 0, \end{aligned}$$

where w_* is given by (4.23). This concludes the proof of Proposition 6.2.

Proposition 6.4 is also a direct consequence of (7.10): instead of computing the limit of

$$\widetilde{\mathcal{I}}_n(\lambda, \eta) = \left[(\widetilde{w}_n^+ - \widetilde{y}_n^+ - \widetilde{y}_n^- + \widetilde{w}_n^-)(\lambda, \eta, \cdot) \right]_n$$

we can compute the limit of

$$2 \left[\widetilde{w}_n^+(\lambda, \eta, \cdot) \right]_n$$

by using very similar arguments.

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